

GORENSTEIN COMPLEXES AND RECOLLEMENTS FROM COTORSION PAIRS

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ABSTRACT. Let R be a Noetherian ring. We find several recollements involving the homotopy categories of various full subcategories of $K(R)$ consisting of complexes built from Gorenstein injective modules. One can be thought of as a Gorenstein version of Krause's recollement $K_{ex}(Inj) \rightarrow K(Inj) \rightarrow \mathcal{D}(R)$. But depending on “how Gorenstein” the ring R is, there may be two other variations of this recollement. We generalize Becker's method from [Bec12] to find these recollements. The result is that we get a theorem that can be applied to find new recollement situations based only on three injective cotorsion pairs related by simple conditions involving class containments. We illustrate this method by pointing out two more recollement situations, again involving Gorenstein complexes. We also look at the ordering on the class of all injective cotorsion pairs on $R\text{-Mod}$ and $\text{Ch}(R)$ and see that the Gorenstein injective cotorsion pair sits at the top of this complete semilattice. There are projective versions of all these results. The analogous projective recollement situations exist whenever R is a coherent ring in which all flat modules have finite projective dimension. The generality of the assumed hypotheses on R is due to recent work in [BGH12].

1. INTRODUCTION

In the paper [Bec12], Becker finds and describes a beautiful way to localize two “injective” abelian model structures to obtain a third abelian model structure. The homotopy categories of the three involved model structures are then linked by a colocalization sequence and the localized model structure is in fact the right Bousfield localization of the first, killing the fibrant objects of the second. Becker goes on to recover Krause's recollement $K_{ex}(Inj) \rightarrow K(Inj) \rightarrow \mathcal{D}(R)$ from [Kra05] using the theory of abelian model categories. Here, $K(Inj)$ is the homotopy category of all complexes of injective modules, $K_{ex}(Inj)$ is the full subcategory of exact complexes of injectives, and $\mathcal{D}(R)$ is the derived category of R . After reading [Bec12] the author recalled seeing in his work instances of injective cotorsion pairs that were “linked” in the same way that allowed Becker to obtain Krause's recollement. One of those situations were the three cotorsion pairs used by Becker to reconstruct the Krause recollement. But another two were Gorenstein injective versions of this. This inspired the author to look for general conditions that allowed for three injective cotorsion pairs to give a recollement. This appears as Theorem 4.6 which we describe in a bit more detail below. The theorem appears quite useful as it allowed the author to not only put together the two Gorenstein injective versions of the recollement alluded to above, but to find an unexpected third variation of the recollement and then without much effort to spot two other recollements involving complexes of Gorenstein modules. These appear in Section 8 and the last Section 9

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but we describe more below. The author expects more recollement situations can either be found or recovered with this method. We now describe in more detail the results in this paper, and the natural separation between Sections 3 through 5 and Sections 6 through 9.

1.1. Weak injective model structures and recollements from cotorsion pairs.

Our main goal in Sections 3 through 5 is to develop a bit more of the theory of injective cotorsion pairs and the localization theory associated to it. The fundamentals of this theory were set in [Bec12] and there is necessarily a bit of overlap here with that paper. Our main additions to this theory are the following: (1) The notion of a weak injective cotorsion pair and a converse and uniqueness statement to Becker’s localization theory from [Bec12]. (2) A general statement for obtaining recollement situations from injective or projective cotorsion pairs. And (3) The maximality property of the Gorenstein injective cotorsion pair.

We now describe these three more. First, we define in Section 3 the notion of an *injective cotorsion pair*. This is a cotorsion pair $(\mathcal{W}, \mathcal{F})$ in any abelian category \mathcal{A} with enough injectives for which \mathcal{W} is *thick* and $\mathcal{W} \cap \mathcal{F}$ is the class of injective objects in \mathcal{A} . As we will make clear, such a cotorsion pair is equivalent to an injective model structure on \mathcal{A} , which as defined in [Gil11] is an abelian model structure on \mathcal{A} in which all objects are cofibrant. Focussing on injective cotorsion pairs themselves is convenient because it puts all of the essential information for an injective model structure in one small package. For example, \mathcal{F} is the class of fibrant objects, \mathcal{W} are the trivial objects, $\mathcal{W} \cap \mathcal{F}$ are the injective (trivially fibrant objects), and the entire model structure is determined from this information. This decluttering allows one to then focus on the relationship between the various classes involved which turns out to be useful for spotting recollement situations.

Becker too considers injective cotorsion pairs in [Bec12], although he did not call them this. Becker shows that if $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ are injective cotorsion pairs (so model structures) with $\mathcal{F}_2 \subseteq \mathcal{F}_1$ then there is a colocalization sequence associated to the homotopy categories $\mathrm{Ho}(\mathcal{M}_2) \xrightarrow{\mathrm{Rid}} \mathrm{Ho}(\mathcal{M}_1) \xrightarrow{\mathrm{Rid}} \mathrm{Ho}(\mathcal{M})$ where $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$ is the right Bousfield localization of \mathcal{M}_1 with respect to killing the class \mathcal{F}_2 of fibrant objects in \mathcal{M}_2 . This is central to this paper and the exact statement reappears in Theorem 3.10. We note two additions in how we state Becker’s result. First we define in Section 3 the notion of a *weak injective* model structure. We point out that Becker’s theorem has a converse in that every hereditary weak injective model structure is the right localization of two injective ones. We also add a uniqueness property that the class of trivial objects satisfy. These two additions are not hard at all but help give a bigger picture to the theory. More importantly, the uniqueness condition is quite useful for actually *spotting* a localization, or even potential recollement situation.

In light of the above, the very rough idea then behind spotting a recollement is to find three injective cotorsion pairs $(\mathcal{W}_1, \mathcal{F}_1)$, $(\mathcal{W}_2, \mathcal{F}_2)$, and $(\mathcal{W}_3, \mathcal{F}_3)$ for which $\mathcal{M}_1/\mathcal{M}_2$ is Quillen equivalent to \mathcal{M}_3 and vice versa $\mathcal{M}_1/\mathcal{M}_3$ is Quillen equivalent to \mathcal{M}_2 . But to get a recollement one needs to “paste” together a localization sequence and a colocalization sequence. So the exact statement we find is this: If $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ and either $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$ OR $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}_3$ then we automatically have an induced recollement situation $\mathcal{F}_2/\sim \rightarrow \mathcal{F}_1/\sim \rightarrow \mathcal{F}_3/\sim$, where $f \sim g$ if and only if $g-f$ factors through an injective object. See Theorem 4.6 for a more detailed statement of the recollement. As we point out in Example 4.8,

the first condition $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$ exactly reflects Becker's idea in [Bec12] to obtain the Krause recollement over a general ring R . We point out in Example 4.8 that this approach does indeed work to recover the Krause recollement for complexes of quasi-coherent sheaves over a general (not necessarily Noetherian) scheme X .

Finally, we describe the maximality property of the Gorenstein injectives and Gorenstein projectives which appears in Section 5. We call an object M in an abelian category with enough injectives *Gorenstein injective* if $M = Z_0 J$ for some exact complex of injectives J which remains exact after applying $\text{Hom}_{\mathcal{A}}(I, -)$ for any other injective object I . The Gorenstein projectives are the dual which we define in an abelian category with enough projectives. It is clear that the canonical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ satisfies $\mathcal{I} \subseteq \mathcal{F}$ whenever $(\mathcal{W}, \mathcal{F})$ is another injective cotorsion pair. We show in Theorem 5.2 two things. First, that for any injective cotorsion pair $(\mathcal{W}, \mathcal{F})$ we have $\mathcal{F} \subseteq \mathcal{GI}$, where \mathcal{GI} is the class of Gorenstein injective objects. Second, whenever the class \mathcal{GI} forms the right half of a complete cotorsion pair $({}^\perp \mathcal{GI}, \mathcal{GI})$, then this is automatically an injective cotorsion pair too. These facts have an immediate application (Theorem 8.1) to a simple characterization of the Gorenstein injective chain complexes in $\text{Ch}(\mathcal{A})$ and lead us to consider a possible lattice structure on \mathcal{A} in Section 6.

1.2. Gorenstein complexes and Gorenstein derived categories. The second main goal of this paper, appearing in Sections 6 through 9, is to build various injective model structures on chain complexes of modules over a ring R based on the Gorenstein injective R -modules. Moreover we wish to apply the above described techniques to obtain recollements between their homotopy categories. Switching to modules over a ring R after working in the general abelian category setting is quite a change and we will comment on possible generalizations to Grothendieck categories a bit more at the end of the introduction. In any case, starting in Section 6 we need to assume more structure on the category other than just abelian because we will need to show that several cotorsion pairs involving chain complexes of Gorenstein injective modules are complete. The standard way to do this is to show that the given cotorsion pair is cogenerated by a set. Or in the language of the more recent [Što11] show that certain classes of complexes are *deconstructible*.

In the recent work of [BGH12] it is shown that the Gorenstein injective cotorsion pair $({}^\perp \mathcal{GI}, \mathcal{GI})$ is a complete cotorsion pair cogenerated by a set whenever R is Noetherian. On the other hand it is shown that the Gorenstein projective cotorsion pair $(\mathcal{GP}, \mathcal{GP}^\perp)$ is a complete cotorsion pair cogenerated by a set whenever R is a coherent ring in which all flat modules have finite projective dimension. See the comments in Subsection 5.1 on the ubiquity of rings in which all flat modules have finite projective dimension. So as described above any (left) Noetherian ring R has a model structure on $R\text{-Mod}$ called the *Gorenstein injective model structure*. It is a generalization of the model structure introduced in [Hov02] for Gorenstein rings. On the other hand, for the (left) coherent rings mentioned above, there is the *Gorenstein projective model structure* on $R\text{-Mod}$ also generalizing one from [Hov02]. To get model structures on $\text{Ch}(R)$ we lift the single injective cotorsion pair $({}^\perp \mathcal{GI}, \mathcal{GI})$ on $R\text{-Mod}$ to several injective model structures on $\text{Ch}(R)$. More generally, in Section 7 we show that *any* injective cotorsion pair on $R\text{-Mod}$ readily lifts to 6, not necessarily different, injective cotorsion pairs (model structures) on $\text{Ch}(R)$. More specifically, given a class of R -modules \mathcal{C} , we have the following classes of chain complexes in $\text{Ch}(R)$.

- $dw\tilde{\mathcal{C}}$ is the class of all chain complexes with $C_n \in \mathcal{C}$.
- $ex\tilde{\mathcal{C}}$ is the class of all exact chain complexes with $C_n \in \mathcal{C}$.
- $\tilde{\mathcal{C}}$ is the class of all exact chain complexes with cycles $Z_n C \in \mathcal{C}$.

Then Theorem 7.2 says the following: Let $(\mathcal{W}, \mathcal{F})$ be an injective cotorsion pair of R -modules which is cogenerated by some set (so a cofibrantly generated model structure on $R\text{-Mod}$). Then below is a list of injective cotorsion pairs in $\text{Ch}(R)$, also each cogenerated by sets (so cofibrantly generated model structures on $\text{Ch}(R)$).

- | | | |
|---|---|--|
| • $({}^\perp dw\tilde{\mathcal{F}}, dw\tilde{\mathcal{F}})$ | $({}^\perp ex\tilde{\mathcal{F}}, ex\tilde{\mathcal{F}})$ | $(dg\tilde{\mathcal{W}}, \tilde{\mathcal{F}})$ |
| • $(dw\tilde{\mathcal{W}}, (dw\tilde{\mathcal{W}})^\perp)$ | $(ex\tilde{\mathcal{W}}, (ex\tilde{\mathcal{W}})^\perp)$ | $(\tilde{\mathcal{W}}, dg\tilde{\mathcal{F}})$ |

In particular, if R is a Noetherian ring then the canonical injective pair $(\mathcal{A}, \mathcal{I})$ along with the Gorenstein injective cotorsion pair $({}^\perp \mathcal{GI}, \mathcal{GI})$ potentially give rise to a great number of model structures on $\text{Ch}(R)$. It is natural to consider any structure that might exist when we order the injective cotorsion pairs $(\mathcal{W}_2, \mathcal{F}_2) \preceq_r (\mathcal{W}_1, \mathcal{F}_1)$ according to $\mathcal{F}_2 \subseteq \mathcal{F}_1$. This ordering corresponds to whether or not the identity functor is a right Quillen functor. We are able to show in Section 6 that the class of all injective cotorsion pairs along with this ordering forms a complete join-semilattice, meaning that suprema exist. However, we don't know if infima exist. As we see in Sections 6 and 7 the number of model structures that we are able to lift to $\text{Ch}(R)$, that is, the number that the author is able to find in the semilattice of injective cotorsion pairs, is directly linked to the global dimension of the ring R that we start with and whether or not it is Gorenstein. So if R is a general non-Gorenstein Noetherian ring, we see numerous injective cotorsion pairs and class containments leading to the recollement situations that appear in Sections 8 and 9.

So why are there three Gorenstein versions of Krause's recollement in the Gorenstein theory? The abstract approach of lifting injective models on $R\text{-Mod}$ to various injective models on $\text{Ch}(R)$ helps answer this question. Consider this: Becker's approach to Krause's recollement reflects the fact that in terms of model categories on $\text{Ch}(R)$, the derived category can be obtained by localizing a model for $K(\text{Inj})$, the homotopy category of complexes of injectives, by a model for the stable derived category $S(R)$ to get a model for $\mathcal{D}(R)$, and that going the other way you can localize a model for $\mathcal{D}(R)$ by a model for $K(\text{Inj})$ to get a model for $S(R)$. However, there is also an analogous trivial recollement. One can take the injective model for $\mathcal{D}(R)$ and localize it by the trivial injective model (having all complexes trivial) to obtain back $\mathcal{D}(R)$. And going the other direction, if you localize the injective model for $\mathcal{D}(R)$ by that same model for $\mathcal{D}(R)$ you get a trivial model structure on $\text{Ch}(R)$. So the answer to the question starting this paragraphs lies in the following observation: When you start with the canonical cotorsion pair $(\mathcal{A}, \mathcal{I})$ in which \mathcal{I} is the class of injectives then the 6 model structures that get lifted to $\text{Ch}(R)$ are not distinct - two of them are trivial. On the other hand, when you start with the Gorenstein injective cotorsion pair $(\mathcal{W}, \mathcal{GI})$ then none of the 6 model structures that get lifted to $\text{Ch}(R)$ are trivial.

1.3. A note on Grothendieck categories. We have chosen at this point to work directly in the category of modules over a ring R and the corresponding category $\text{Ch}(R)$ of chain complexes rather than attempt to obtain results for a general Grothendieck category. The theory for modules over a ring is of course simpler but this helps us make some of the basic points we have described above. However, recent strides have been made, for example in [SŠ11] and [Što11], which help us to

understand much better deconstructible classes in Grothendieck categories. Therefore, similar versions of the results in Sections 6 through 9 should be obtainable for a general Grothendieck category \mathcal{G} . But of course these categories need not have projective generators and this causes a problem. One can get around this problem by assuming we have a generating set for \mathcal{G} which is trivial in whichever injective model structure we are considering. That is, we need to assume there is a generator G for \mathcal{G} for which $G \in \mathcal{W}$ whenever $(\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair. Surely there are some schemes X which satisfy this, but it is not clear to the author just how often this is the case. The author hopes to return to this in the future. We point out again that the first five sections of this paper are described in great generality. In particular, the recollement situations will hold at once after one proves the existence of the proper cotorsion pairs (or model structures) in a more general setting.

The outline of the paper is as follows. Preliminary notions and notation are laid out in Section 2. In Section 3 we look at the necessary notions of Hovey triples, injective cotorsion pairs, weak injective cotorsion pairs, and Becker's localization theorem. In Section 4 we discuss recollements and how they can be obtained from cotorsion pairs. In Section 5 we prove the maximality property of the Gorenstein injective pair. In Section 6 we look at the semilattice of injective cotorsion pairs. Then in Section 7 we show how to lift an injective cotorsion pairs from $R\text{-Mod}$ to six in $\text{Ch}(R)$. Finally in Section 8 we look exclusively at the Gorenstein complexes and the Krause-like recollements that come up. In Section 9 we point out two more recollement situations.

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2. PRELIMINARIES

Here we either recall or define fundamental ideas and set some notation which we use throughout the paper. By a ring R , we always mean a ring with 1. Central to this paper is Quillen's notion of a model category [Qui67]. Our basic reference is [Hov99] and we are concerned with abelian model structures. The basic theory of abelian model structures is in [Hov02], but we will recall the correspondence with cotorsion pairs in this section below.

2.1. Chain complexes. We denote by $R\text{-Mod}$ the category of (left) R -modules and by $\text{Ch}(R)$ the category of chain complexes of (left) R -modules. We write chain complexes as $\cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots$ so that the differentials lower the degree.

We let $S^n(M)$ denote the chain complex with all entries 0 except M in degree n . We let $D^n(M)$ denote the chain complex X with $X_n = X_{n-1} = M$ and all other entries 0. All maps are 0 except $d_n = 1_M$. Given X , the *suspension of X* , denoted ΣX , is the complex given by $(\Sigma X)_n = X_{n-1}$ and $(d_{\Sigma X})_n = -d_n$. The complex $\Sigma(\Sigma X)$ is denoted $\Sigma^2 X$ and inductively we define $\Sigma^n X$ for all $n \in \mathbb{Z}$.

Given two chain complexes X and Y we define $\text{Hom}(X, Y)$ to be the complex of abelian groups $\cdots \rightarrow \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n-1}) \rightarrow \cdots$,

where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. This gives a functor $\text{Hom}(X, -): \text{Ch}(R) \rightarrow \text{Ch}(\mathbb{Z})$ which is left exact, and exact if X_n is projective for all n . Similarly the contravariant functor $\text{Hom}(-, Y)$ sends right exact sequences to left exact sequences and is exact if Y_n is injective for all n .

Recall that $\text{Ext}_{\text{Ch}(R)}^1(X, Y)$ is the group of (equivalence classes) of short exact sequences $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ under the Baer sum. We let $\text{Ext}_{dw}^1(X, Y)$ be the subgroup of $\text{Ext}_{\text{Ch}(R)}^1(X, Y)$ consisting of those short exact sequences which are split in each degree. We often make use of the following standard fact.

Lemma 2.1. *For chain complexes X and Y , we have*

$$\text{Ext}_{dw}^1(X, \Sigma^{(-n-1)} Y) \cong H_n \text{Hom}(X, Y) = \text{Ch}(R)(X, \Sigma^{-n} Y) / \sim,$$

where \sim is chain homotopy.

In particular, for chain complexes X and Y , $\text{Hom}(X, Y)$ is exact iff for any $n \in \mathbb{Z}$, any $f: \Sigma^n X \rightarrow Y$ is homotopic to 0 (or iff any $f: X \rightarrow \Sigma^n Y$ is homotopic to 0).

2.2. Cotorsion pairs. The most important concept in this paper is that of a cotorsion pair in an abelian category \mathcal{A} . A standard reference is [EJ00].

Definition 2.2. A pair of classes $(\mathcal{F}, \mathcal{C})$ in an abelian category \mathcal{A} is a cotorsion pair if the following conditions hold:

- (1) $\text{Ext}_{\mathcal{A}}^1(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$.
- (2) If $\text{Ext}_{\mathcal{A}}^1(F, X) = 0$ for all $F \in \mathcal{F}$, then $X \in \mathcal{C}$.
- (3) If $\text{Ext}_{\mathcal{A}}^1(X, C) = 0$ for all $C \in \mathcal{C}$, then $X \in \mathcal{F}$.

A cotorsion pair is said to have *enough projectives* if for any $X \in \mathcal{A}$ there is a short exact sequence $0 \rightarrow C \rightarrow F \rightarrow X \rightarrow 0$ where $C \in \mathcal{C}$ and $F \in \mathcal{F}$. We say it has *enough injectives* if it satisfies the dual statement. If both of these hold we say the cotorsion pair is *complete*. Whenever the category \mathcal{A} has enough injectives and projectives then a cotorsion pair $(\mathcal{F}, \mathcal{C})$ is complete iff $(\mathcal{F}, \mathcal{C})$ has enough injectives iff $(\mathcal{F}, \mathcal{C})$ has enough projectives. [EJ00, Proposition 7.1.7].

In $R\text{-Mod}$, the class of projectives is the left half of an obvious complete cotorsion pair while the class of injectives is the right half of an obvious complete cotorsion pair. The most well-known nontrivial example of a complete cotorsion pair is $(\mathcal{F}, \mathcal{C})$ where \mathcal{F} is the class of flat modules and \mathcal{C} are the cotorsion modules. A proof that this is a complete cotorsion theory can be found in [EJ00]. We will be concerned with cotorsion pairs of chain complexes in this paper and will use results from [Gil04] and [Gil08]. These results describe several cotorsion pairs in $\text{Ch}(R)$ that can be associated to a single given cotorsion pair in $R\text{-Mod}$. This will be encountered in Section 7 where we will recall the basic definitions.

2.3. Hereditary cotorsion pairs. Let \mathcal{A} be an abelian category. We say a cotorsion pair $(\mathcal{F}, \mathcal{C})$ in \mathcal{A} is *resolving* if \mathcal{F} is closed under taking kernels of epimorphisms between objects of \mathcal{F} . That is, if for any short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, we have X in \mathcal{F} whenever Y and Z are in \mathcal{F} . We say $(\mathcal{F}, \mathcal{C})$ is *coresolving* if the right hand class \mathcal{C} satisfies the dual. Finally, we say that $(\mathcal{F}, \mathcal{C})$ is *hereditary* if it is both resolving and coresolving. The following is a standard test for checking to see if a given cotorsion pair is hereditary.

Lemma 2.3 (Hereditary Test). *Assume $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in an abelian category \mathcal{A} . Consider the statements below.*

- (1) \mathcal{F} is closed under taking kernels of epimorphisms between objects of \mathcal{F} .
- (2) \mathcal{F} is syzygy closed, meaning $X \in \mathcal{F}$ whenever we have an exact $0 \rightarrow X \rightarrow P \rightarrow Z \rightarrow 0$ with P projective and $Z \in \mathcal{F}$.
- (3) \mathcal{C} is closed under taking cokernels of monomorphisms between objects of \mathcal{C} .
- (4) \mathcal{C} is cosyzygy closed, meaning $Z \in \mathcal{C}$ whenever we have an exact $0 \rightarrow X \rightarrow I \rightarrow Z \rightarrow 0$ with I injective and $X \in \mathcal{C}$.
- (5) $\text{Ext}_{\mathcal{A}}^2(F, C) = 0$ whenever $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

Then we have the following implications depending on whether or not \mathcal{A} has enough injectives or projectives.

- *If \mathcal{A} has enough injectives then we have (3) implies (4) implies (5) implies (1) implies (2).*
- *If \mathcal{A} has enough projective then we have (1) implies (2) implies (5) implies (3) implies (4).*
- *(5) implies (1) – (4) without any assumption on enough injectives or projectives. (Just using the long exact sequence in Ext .)*

In particular, when \mathcal{A} is a category with enough projectives and injectives then all of the conditions (1) – (5) are equivalent.

Proof. See [GR99]. □

So suppose \mathcal{A} has enough injectives and you have a cotorsion pair $(\mathcal{F}, \mathcal{C})$. Then from Lemma 2.3 the cotorsion pair is hereditary whenever it is known to just be coresolving, or even if \mathcal{C} is just cosyzygy closed. However it can't be used to conclude hereditary in the case that you only know it is resolving. But for a complete cotorsion pair, the following is a wonderfully convenient result that appears as Corollary 1.1.13 of [Bec12].

Lemma 2.4 (Becker's Lemma). *Let \mathcal{A} be an abelian category and let $(\mathcal{F}, \mathcal{C})$ be a complete cotorsion pair. Then the following are equivalent.*

- (1) $(\mathcal{F}, \mathcal{C})$ is hereditary.
- (2) $(\mathcal{F}, \mathcal{C})$ is resolving.
- (3) $(\mathcal{F}, \mathcal{C})$ is coresolving.

2.4. Hovey's correspondence. Hovey defines abelian model categories in [Hov02]. He then characterizes them in terms of cotorsion pairs as we now describe. So in fact one could even take the cotorsion pairs given in the correspondence below as the definition of an abelian model category. First, we need the definition of a thick subcategory.

Definition 2.5. By a *thick subcategory* of an abelian \mathcal{A} we mean a class of objects \mathcal{W} which is closed under direct summands and such that if two out of three of the terms in a short exact sequence are in \mathcal{W} , then so is the third.

Theorem 2.6. *Let \mathcal{A} be an abelian category with an abelian model structure. Let \mathcal{Q} be the class of cofibrant objects, \mathcal{R} the class of fibrant objects and \mathcal{W} the class of trivial objects. Then \mathcal{W} is a thick subcategory of \mathcal{A} and both $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs in \mathcal{A} . Conversely, given a thick subcategory \mathcal{W} and classes \mathcal{Q} and \mathcal{R} making $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ each complete cotorsion*

pairs, then there is an abelian model structure on \mathcal{A} where \mathcal{Q} are the cofibrant objects, \mathcal{R} are the fibrant objects and \mathcal{W} are the trivial objects.

We point out that the abelian model structure on \mathcal{A} is then completely determined by the classes \mathcal{Q} , \mathcal{W} and \mathcal{R} . Indeed the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernel in \mathcal{Q} (resp. $\mathcal{Q} \cap \mathcal{W}$) and the fibrations are the epimorphisms with kernel in \mathcal{R} (resp. $\mathcal{W} \cap \mathcal{R}$). The weak equivalences are the maps which factor as a trivial cofibration followed by a trivial fibration. However, the description of the weak equivalences given by the Lemma below is sometimes more convenient. It follows from [Hov02, Lemma 5.8] and an application of the two out of three axiom.

Lemma 2.7. *Say \mathcal{A} is an abelian category with an abelian model structure. Let \mathcal{W} denote the class of trivial objects. Then a map f is a weak equivalence if and only if it factors as a monomorphism with cokernel in \mathcal{W} followed by an epimorphism with kernel in \mathcal{W} .*

Remark 1. We note that one normally assumes the category \mathcal{A} is bicomplete for it to qualify as a model category. However, as explained in [Gil11] an abelian category with an abelian model structure already has enough colimits and limits to get the basic first results of homotopy theory. In circumstances where we do not explicitly state that \mathcal{A} is bicomplete we will deliberately use the term model *structure* on \mathcal{A} and only say that \mathcal{A} is a model *category* when \mathcal{A} is bicomplete.

3. INJECTIVE AND WEAK INJECTIVE HOVEY TRIPLES

It was reading [Bec12] that led the author to denote abelian model structures as triples and this eventually led to the definition of a weak injective model structure as below. Throughout this entire section assume \mathcal{A} is an abelian category.

3.1. Hovey triples. We start with a convenient definition.

Definition 3.1. Let \mathcal{Q} , \mathcal{W} , and \mathcal{R} be three classes in \mathcal{A} as in Hovey's correspondence. Then we call $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ a *Hovey triple*. By a *hereditary* Hovey triple we mean that the two corresponding cotorsion pairs $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are each hereditary.

Notation. Second, we note that due to Hovey's one-to-one correspondence between Hovey triples in \mathcal{A} and abelian model structures on \mathcal{A} we often will not distinguish between the Hovey triple and the actual model structure. For example, we may say $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is an abelian model structure and understand this to mean the model structure associated to the Hovey triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$. On the other hand, we may say that an abelian model structure is *hereditary* and by this we mean that the Hovey triple is hereditary.

3.2. Injective and weak injective Hovey triples.

Definition 3.2. We call a Hovey triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ *injective* if $\mathcal{Q} = \mathcal{A}$. That is, if every object of \mathcal{A} is cofibrant. By a *weak injective* Hovey triple we mean a Hovey triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ for which $\mathcal{Q} \cap \mathcal{W} \cap \mathcal{R}$ is exactly the class of injective objects in \mathcal{A} . We have the obvious dual notions of *projective* Hovey triples and *weak projective* Hovey triples.

We apply these same phrases to the actual model structures induced as well. So we speak of *injective*, *weak injective*, *projective*, and *weak projective* model structures on \mathcal{A} .

Remark 2. It is immediate that an injective Hovey triple is a weak injective Hovey triple. Also an injective Hovey triple is automatically hereditary by Becker's Lemma 2.4 because \mathcal{W} is thick.

Note that whenever $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is an injective Hovey triple then the class \mathcal{Q} is redundant and need not be mentioned. Indeed as long as \mathcal{A} has enough injectives, then an injective model structure on \mathcal{A} is equivalent to a single complete cotorsion pair $\mathcal{M} = (\mathcal{W}, \mathcal{R})$ where \mathcal{W} is thick and $\mathcal{W} \cap \mathcal{R}$ is the class of injective objects. We make this precise in the next definition.

Definition 3.3. Let \mathcal{A} be an abelian model category with enough injectives. We call a complete cotorsion pair $(\mathcal{W}, \mathcal{F})$ an *injective cotorsion pair* if \mathcal{W} is thick and $\mathcal{W} \cap \mathcal{F}$ coincides with the class of injective objects. In this case $\mathcal{M} = (\mathcal{W}, \mathcal{F})$ determines an injective model structure on \mathcal{A} with \mathcal{F} the class of fibrant objects and \mathcal{W} the trivial objects. On the other hand, if \mathcal{A} has enough projectives we define the dual notion of a *projective cotorsion pair* $(\mathcal{C}, \mathcal{W})$.

Remark 3. We emphasize that we don't use the term *injective cotorsion pair* unless \mathcal{A} has enough injectives and we don't use the phrase *projective cotorsion pair* unless \mathcal{A} has enough projectives. The point is that a projective cotorsion pair IS a model structure on \mathcal{A} but you don't have a model structure if \mathcal{A} lacks projectives. The definitions are useful though because now all of the essential information needed for such a model structure is packed into one simple idea - a single cotorsion pair.

The definition of an injective cotorsion pair is stronger than what we need. This isn't just interesting but it is relevant for stating a converse to Becker's localization theorem (Theorem 3.10). The author learned the Lemma below from Henrik Holm.

Lemma 3.4. *Assume $(\mathcal{W}, \mathcal{F})$ is an hereditary cotorsion pair in \mathcal{A} . Suppose that for any $F \in \mathcal{F}$ we can find a short exact sequence $0 \rightarrow F' \rightarrow I \rightarrow F \rightarrow 0$ where I is injective and $F' \in \mathcal{F}$. Then \mathcal{W} is thick.*

Proof. The assumptions imply that \mathcal{W} is already closed under retracts and extensions and is resolving. So it remains to show that that if

$$(*) \quad 0 \rightarrow V \rightarrow W \rightarrow X \rightarrow 0$$

is an exact sequence with $V, W \in \mathcal{W}$ then $X \in \mathcal{W}$ too. Applying $\text{Hom}_{\mathcal{A}}(-, F)$ to $(*)$, for any $F \in \mathcal{F}$, it follows that $\text{Ext}_{\mathcal{A}}^{\geq 2}(X, F) = 0$. To see that $\text{Ext}_{\mathcal{A}}^1(X, F) = 0$ for every $F \in \mathcal{F}$, pick a short exact sequence $0 \rightarrow F' \rightarrow I \rightarrow F \rightarrow 0$, where I is injective and $F' \in \mathcal{F}$. Applying $\text{Hom}_{\mathcal{A}}(X, -)$ to this sequence gives $\text{Ext}_{\mathcal{A}}^1(X, F) \cong \text{Ext}_{\mathcal{A}}^2(F', X)$, which is zero by what we just proved. \square

Theorem 3.5 (Characterizations of injective cotorsion pairs). *Suppose $(\mathcal{W}, \mathcal{F})$ is a complete cotorsion pair in an abelian category \mathcal{A} with enough injectives. Then each of the following statements are equivalent:*

- (1) $(\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair.
- (2) $(\mathcal{W}, \mathcal{F})$ is hereditary and $\mathcal{W} \cap \mathcal{F}$ equals the class of injective objects.
- (3) \mathcal{W} is thick and contains the injective objects.

Proof. First, (1) implies (2) by definition and Becker's Lemma 2.4. For (2) implies (3) note that for any $F \in \mathcal{F}$ using that the cotorsion pair $(\mathcal{W}, \mathcal{F})$ has enough projectives we can write $0 \rightarrow F' \rightarrow W \rightarrow F \rightarrow 0$ where $W \in \mathcal{W}$ and $F' \in \mathcal{F}$. Then since \mathcal{F} is closed under extensions we have $W \in \mathcal{F}$. So by hypothesis we have W is injective. Therefore (2) implies (3) follows from Lemma 3.4.

For (3) implies (1) first note that the hypothesis makes clear that the injective objects are in $\mathcal{W} \cap \mathcal{F}$. So we just wish to show that everything in $\mathcal{W} \cap \mathcal{F}$ is injective. So suppose $X \in \mathcal{W} \cap \mathcal{F}$. Then using that \mathcal{A} has enough injectives find a short exact sequence $0 \rightarrow X \rightarrow I \rightarrow I/X \rightarrow 0$ where I is injective. By hypothesis $I \in \mathcal{W}$ and also \mathcal{W} is assumed to be thick, which means $I/X \in \mathcal{W}$. But now since $(\mathcal{W}, \mathcal{F})$ is a cotorsion pair the exact sequence splits. Therefore X is a direct summand of I , proving $X \in \mathcal{I}$. \square

By duality we have the following as well.

Theorem 3.6 (Characterizations of projective cotorsion pairs). *Suppose $(\mathcal{C}, \mathcal{W})$ is a complete cotorsion pair in an abelian category \mathcal{A} with enough projectives. Then each of the following statements are equivalent:*

- (1) $(\mathcal{C}, \mathcal{W})$ is a projective cotorsion pair.
- (2) $(\mathcal{C}, \mathcal{W})$ is hereditary and $\mathcal{C} \cap \mathcal{W}$ equals the class of projective objects.
- (3) \mathcal{W} is thick and contains the projective objects.

For a general Grothendieck category \mathcal{A} it becomes important to know whether or not the left side of a cotorsion pair contains a set of generators for \mathcal{A} . We point out the following property of injective cotorsion pairs which could be of use in this setting. It won't be used in this paper however.

Proposition 3.7. *Let \mathcal{A} be an abelian category with enough injectives. If $(\mathcal{W}, \mathcal{F})$ is any injective cotorsion pair, then \mathcal{W} contains all objects of finite injective dimension. If \mathcal{A} also has enough projectives then \mathcal{W} also contains all objects of finite projective dimension.*

We have the dual for projective cotorsion pairs $(\mathcal{C}, \mathcal{W})$.

Proof. The point is that \mathcal{W} is thick and contains all injective and projective objects. So let M be of finite injective dimension. Then there exists an exact sequence

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^n \rightarrow 0$$

with each I^n injective. Since each $I^i \in \mathcal{W}$ and \mathcal{W} thick we conclude $M \in \mathcal{I}$. If \mathcal{A} has enough projectives, then the same argument works for projective dimension since projectives are always in the left side of a cotorsion pair. \square

Note that Proposition 3.7 is saying that any object of finite injective dimension is trivial in the model structure associated to any injective cotorsion pair.

3.3. The homotopy category of a weak injective model structure. It is convenient to explicitly define weak injective model structures for two reasons. First, the hereditary ones are precisely the right Bousfield localizations of two injective ones as we point out in Theorem 3.10. Second the homotopy relation on the subcategory of cofibrant-fibrant objects is characterized in a very nice way which we now explain. This will prove important for this paper. For example, it is automatic that for any weak injective (or weak projective) model structure on $\text{Ch}(R)$, two

maps from a cofibrant complex to a fibrant complex are formally homotopic if and only if they are chain homotopic in the usual sense.

Proposition 3.8. *Suppose $\mathcal{M} = (\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair in \mathcal{A} .*

- (1) *If Y is fibrant then two maps $f, g: X \rightarrow Y$ in \mathcal{A} are homotopic, written $f \sim g$, if and only if $g - f$ factors through an injective object.*
- (2) *The inclusion $i: \mathcal{F} \rightarrow \mathcal{A}$ of the fully exact subcategory of fibrant objects \mathcal{F} induces an inclusion of categories $\mathrm{Ho} i: \mathcal{F}/\sim \rightarrow \mathrm{Ho}(\mathcal{A})$ displaying \mathcal{F}/\sim as a full equivalent subcategory of $\mathrm{Ho}(\mathcal{A})$.*
- (3) *The inverse of $\mathrm{Ho} i$ is the functor $\mathrm{Ho} R: \mathrm{Ho}(\mathcal{A}) \rightarrow \mathcal{F}/\sim$ and this is the fibrant replacement functor.*

Proposition 3.9. *Suppose $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is an hereditary weak injective model structure in \mathcal{A} .*

- (1) *If X is cofibrant and Y is fibrant then two maps $f, g: X \rightarrow Y$ in \mathcal{A} are homotopic, written $f \sim g$, if and only if $g - f$ factors through an injective object.*
- (2) *The inclusion $i: \mathcal{Q} \cap \mathcal{R} \rightarrow \mathcal{A}$ of the fully exact subcategory of cofibrant-fibrant objects induces an inclusion of categories $\mathrm{Ho} i: \mathcal{Q} \cap \mathcal{R}/\sim \rightarrow \mathrm{Ho}(\mathcal{A})$ displaying $\mathcal{Q} \cap \mathcal{R}/\sim$ as a full equivalent subcategory of $\mathrm{Ho}(\mathcal{A})$.*
- (3) *The inverse of $\mathrm{Ho} i$ is the functor $\mathrm{Ho} Q \circ \mathrm{Ho} R: \mathrm{Ho}(\mathcal{A}) \rightarrow \mathcal{Q} \cap \mathcal{R}/\sim$ and this is fibrant replacement followed by cofibrant replacement.*

Proof. Both follow immediately from [Gil11, Proposition 4.4 and Section 5]. □

3.4. Becker's Theorem and a converse. We now look at a beautiful result from [Bec12, Proposition 1.4.2] giving a simple description, in terms of Hovey triples, of the right Bousfield localization of an injective model structure with respect to another. In our statement here we add a uniqueness property and a converse. These last two things are not hard at all but they give a complete picture and are in fact useful for spotting localizations. In fact the author had already witnessed on more than one occasion model structures on $\mathrm{Ch}(R)$ arise in the following way. First, let $(\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be two injective model structures on \mathcal{A} . Suppose you also have a thick subcategory \mathcal{W} for which $\mathcal{M} = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$ is a Hovey triple. Then these three model structures are obviously linked in some way. It was not until the result of Becker that the author learned the formal connection. It turns out that $\mathcal{M} = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$ is the right Bousfield localization of \mathcal{M}_1 with respect to the fibrant objects in \mathcal{M}_2 .

Theorem 3.10 (Characterization of Becker Localizations). *Let \mathcal{A} be a bicomplete abelian category with enough injectives. Let $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be two injective cotorsion pairs on \mathcal{A} with $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Then there exists a weak injective hereditary Hovey triple $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$ on \mathcal{A} where the thick class \mathcal{W} is*

$$\begin{aligned} \mathcal{W} &= \{ X \in \mathcal{A} \mid \exists \text{ s.e.s. } 0 \rightarrow X \rightarrow F_2 \rightarrow W_1 \rightarrow 0 \text{ with } F_2 \in \mathcal{F}_2, W_1 \in \mathcal{W}_1 \} \\ &= \{ X \in \mathcal{A} \mid \exists \text{ s.e.s. } 0 \rightarrow F_2 \rightarrow W_1 \rightarrow X \rightarrow 0 \text{ with } F_2 \in \mathcal{F}_2, W_1 \in \mathcal{W}_1 \} \end{aligned}$$

We call $\mathcal{M}_1/\mathcal{M}_2$ the right localization of \mathcal{M}_1 with respect to \mathcal{M}_2 and we note the following uniqueness and converse:

- (1) (Uniqueness of Trivial Objects) *Suppose \mathcal{V} is a thick subcategory for which $\mathcal{M} = (\mathcal{W}_2, \mathcal{V}, \mathcal{F}_1)$ is a Hovey triple. Then $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$.*
- (2) (Converse) *Let $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ be a weak injective hereditary Hovey triple in \mathcal{A} . Then setting $\mathcal{M}_1 = (\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $\mathcal{M}_2 = (\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$, these are each injective cotorsion pairs and $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$.*

Remark 4. Becker shows that his right localization $\mathcal{M}_1/\mathcal{M}_2$ is in fact the right Bousfield localization with respect to the maps $0 \rightarrow F$ where $F \in \mathcal{F}_2$. We note also that we have assumed here for the first time that \mathcal{A} is bicomplete. This is really not necessary but it is harmless since in all the applications we have in mind \mathcal{A} will be a bicomplete abelian model category. See Remark 1 which reflects the philosophy that all model categories ought to be bicomplete anyway.

Proof. This is Becker's result. See [Bec12, Proposition 1.4.2]. We are simply noting the uniqueness and converse statements.

(Uniqueness of Trivial Objects.) Note that by the definition of $(\mathcal{W}_2, \mathcal{V}, \mathcal{F}_1)$ being a Hovey triple we have $(\mathcal{W}_2 \cap \mathcal{V}, \mathcal{F}_1)$ and $(\mathcal{W}_2, \mathcal{V} \cap \mathcal{F}_1)$ are each cotorsion pairs. It follows immediately that $\mathcal{W}_2 \cap \mathcal{V} = \mathcal{W}_1$ and $\mathcal{V} \cap \mathcal{F}_1 = \mathcal{F}_2$. To see that $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$, all that is required is to show

$$\mathcal{V} = \{ X \in \mathcal{A} \mid \exists \text{ s.e.s. } 0 \rightarrow X \rightarrow F_2 \rightarrow W_1 \rightarrow 0 \text{ with } F_2 \in \mathcal{F}_2, W_1 \in \mathcal{W}_1 \}.$$

So let $X \in \mathcal{V}$. Then using that $(\mathcal{W}_1, \mathcal{F}_1)$ has enough injectives we get a short exact sequence $0 \rightarrow X \rightarrow F_1 \rightarrow W_1 \rightarrow 0$. Since \mathcal{V} is thick it is closed under extensions and so from what we noted above we conclude that $F_1 \in \mathcal{V} \cap \mathcal{F}_1 = \mathcal{F}_2$. So X is contained in the class

$$\{ X \in \mathcal{A} \mid \exists \text{ s.e.s. } 0 \rightarrow X \rightarrow F_2 \rightarrow W_1 \rightarrow 0 \text{ with } F_2 \in \mathcal{F}_2, W_1 \in \mathcal{W}_1 \}.$$

On the other hand, if X fits in such a short exact sequence $0 \rightarrow X \rightarrow F_2 \rightarrow W_1 \rightarrow 0$ then since $F_2, W_1 \in \mathcal{V}$ we also get $X \in \mathcal{V}$ by thickness.

(Converse.) Let $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ be any weak injective and hereditary Hovey triple in \mathcal{A} . By definition of $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ being hereditary we have that each of the cotorsion pairs $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ are also hereditary. Setting $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1) = (\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2) = (\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ we get that \mathcal{M}_1 and \mathcal{M}_2 are each injective cotorsion pairs by condition (2) of Theorem 3.5. It now follows that $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$ by the uniqueness property of the trivial objects that we just prove above. \square

Example 3.11. It is easy to see that $\mathcal{M} = (\mathcal{W}, \mathcal{F})$ is an injective model structure if and only if it is the right localization of itself by the trivial model structure induced by the categorical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$.

Example 3.12. Let $\mathcal{A} = \text{Ch}(R)$. Let $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ be the Inj model structure from [BGH12] which has as the fibrant complexes $\mathcal{F}_1 = dw\tilde{\mathcal{I}}$, the class of all complexes of injective R -modules. Also let $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be the exact Inj model structure from [BGH12] which has as the fibrant complexes $\mathcal{F}_2 = ex\tilde{\mathcal{I}}$, the class of all exact complexes of injective R -modules. Denote the class of exact complexes by \mathcal{E} . Then it follows from Theorem 4.7 of [Gil08] that there is a hereditary weak injective Hovey triple $\mathcal{M} = (\mathcal{W}_2, \mathcal{E}, \mathcal{F}_1)$. It follows that $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$. Its homotopy

category is $\mathcal{D}(R)$ because the trivial complexes are precisely the exact complexes (and so it follows from Lemma 2.7 that the homology isomorphisms are the weak equivalences). This example is relevant to Becker's approach to recovering Krause's recollement. We see Gorenstein injective analogs in Section 8.

Remark 5. We have worked with injective cotorsion pairs and their right localizations in this section. We note that the dual statements concerning left localizations of projective cotorsion pairs also hold. We have omitted the projective statements in this section.

4. LOCALIZATION SEQUENCES AND RECOLLEMENTS FROM COTORSION PAIRS

The goal here is to describe how a recollement can be obtained from three interrelated injective cotorsion pairs. The method is a generalization of Becker's approach from [Bec12] where he obtained Krause's recollement from [Kra05], for a general ring R , using the theory of model categories. We start with the definition of a recollement. Loosely, a recollement is an "attachment" of two triangulated categories. The standard reference is [BBD82]. We follow [Kra05, Sections 3 and 4].

We note that the homotopy category of an abelian model category is always a triangulated category and has a set of weak generators whenever the model structure is cofibrantly generated. See Hovey [Hov02, Section 7] for more details.

First, we give the definition of a localization sequence that appeared in [Kra05].

Definition 4.1. Let $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ be a sequence of exact functors between triangulated categories. We say it is a *localization sequence* when there exists right adjoints F_ρ and G_ρ giving a diagram of functors as below with the listed properties.

$$\mathcal{T}' \xrightleftharpoons[F_\rho]{F} \mathcal{T} \xrightleftharpoons[G_\rho]{G} \mathcal{T}''$$

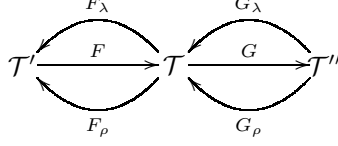
- (1) The right adjoint F_ρ of F satisfies $F_\rho \circ F \cong \text{id}_{\mathcal{T}'}$.
- (2) The right adjoint G_ρ of G satisfies $G \circ G_\rho \cong \text{id}_{\mathcal{T}''}$.
- (3) For any object $X \in \mathcal{T}$, we have $GX = 0$ iff $X \cong FX'$ for some $X' \in \mathcal{T}'$.

A *colocalization sequence* is the dual. That is, there must exist left adjoints F_λ and G_λ with the analogous properties.

It is fair to say that a localization sequence is a sequence of left adjoints which in some sense "splits" at the level of triangulated categories. See [Kra06, Section 3] for the first properties of localization sequences which reflect this statement. Similarly, a colocalization sequence is a sequence of right adjoints with this property. It is true that if $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ is a localization sequence then $\mathcal{T}'' \xrightarrow{G_\rho} \mathcal{T} \xrightarrow{F_\rho} \mathcal{T}'$ is a colocalization sequence and if $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ is a colocalization sequence then $\mathcal{T}'' \xrightarrow{G_\lambda} \mathcal{T} \xrightarrow{F_\lambda} \mathcal{T}'$ is a localization sequence. This brings us to the definition of a recollement where the sequence of functors $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ is both a localization sequence and a colocalization sequence.

Definition 4.2. Let $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ be a sequence of exact functors between triangulated categories. We say $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ induces a *recollement* if it is both

a localization sequence and a colocalization sequence as shown in the picture.



So the idea will be to “glue” a colocalization sequence to a localization sequence to get the diagram of functors in the recollement diagram.

4.1. Colocalization sequences from weak injective model structures. Lets look again at the setup to Theorem 3.10. We are working in a bicomplete abelian category \mathcal{A} with enough injectives. We have two injective cotorsion pairs $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ with $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Since the identity functor is exact it follows immediately that $\mathcal{M}_2 \xrightarrow{\text{id}} \mathcal{M}_1 \xrightarrow{\text{id}} \mathcal{M}_1/\mathcal{M}_2$ are right Quillen functors. That is, we have Quillen adjunctions $\mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_1 \rightleftarrows \mathcal{M}_2$ consisting entirely of identity functors. The following comes from [Bec12, Corollary 1.4.5].

Proposition 4.3. *Let \mathcal{A} be a bicomplete abelian category with enough injectives and let $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be two injective cotorsion pairs on \mathcal{A} with $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Then the identity Quillen adjunctions $\mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_1 \rightleftarrows \mathcal{M}_2$ descend to a colocalization sequence $\text{Ho}(\mathcal{M}_2) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1/\mathcal{M}_2)$ with left adjoints Lid . In particular, on the level of the full subcategory of cofibrant-fibrant subobjects we have the colocalization sequence:*

$$\mathcal{F}_2/\sim \xleftarrow[I]{E(\mathcal{M}_2)} \mathcal{F}_1/\sim \xleftarrow[C(\mathcal{M}_2)]{I} (\mathcal{F}_1 \cap \mathcal{W}_2)/\sim$$

Here the functors I are each inclusion, the functor $E(\mathcal{M}_2)$ represents using enough injectives of the cotorsion pair $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ (to get what would often be called a special \mathcal{F}_2 -preenvelope), and the functor $C(\mathcal{M}_2)$ represents using enough projectives of the cotorsion pair $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ (to get what would often be called a special- \mathcal{W}_2 precover).

Proof. See [Bec12, Corollary 1.4.5] to see that we have the colocalization sequence $\text{Ho}(\mathcal{M}_2) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1/\mathcal{M}_2)$. In general, the right derived functor is defined on objects by first taking a fibrant replacement and then applying the functor, here the identity. Similarly, the left derived functor is defined by first taking a cofibrant replacement and then applying the functor. In any abelian model category $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$, taking cofibrant replacements corresponds to using enough projectives of the cotorsion pair $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ and taking fibrant replacements corresponds to using enough injectives of the cotorsion pair $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$. Also as in Propositions 3.8 and 3.9 one uses cofibrant-fibrant replacement and inclusion when translating between the homotopy categories of \mathcal{M} and the full subcategory of cofibrant-fibrant objects. We conclude that the functors work as stated. \square

4.2. Finding a recollement from three cotorsion pairs. With the same setup as in Subsection 4.1, now suppose we have three injective model structures $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$, $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$, and $\mathcal{M}_3 = (\mathcal{W}_3, \mathcal{F}_3)$ having $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$. Then

for $i = 1, 2$ we have from Proposition 4.3 that the identity functors give Quillen adjunctions $\mathcal{M}_1/\mathcal{M}_i \rightleftarrows \mathcal{M}_1 \rightleftarrows \mathcal{M}_i$ descending to give two colocalization sequences

$$(*) \quad \mathrm{Ho}(\mathcal{M}_2) \xrightarrow{\mathrm{Rid}} \mathrm{Ho}(\mathcal{M}_1) \xrightarrow{\mathrm{Rid}} \mathrm{Ho}(\mathcal{M}_1/\mathcal{M}_2)$$

and

$$(**) \quad \mathrm{Ho}(\mathcal{M}_3) \xrightarrow{\mathrm{Rid}} \mathrm{Ho}(\mathcal{M}_1) \xrightarrow{\mathrm{Rid}} \mathrm{Ho}(\mathcal{M}_1/\mathcal{M}_3).$$

However, we also have the following lemma.

Lemma 4.4. *The identity adjunctions shown below are in fact Quillen adjunction:*

$$\mathcal{M}_3 \rightleftarrows \mathcal{M}_3, \quad \mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_3, \quad \mathcal{M}_1/\mathcal{M}_3 \rightleftarrows \mathcal{M}_2$$

Proof. The first is obvious and the second two are symmetric, so we show that $\mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_3$ is a Quillen adjunction. That is, the identity from \mathcal{M}_3 to $\mathcal{M}_1/\mathcal{M}_2$ is right Quillen. For this just recall $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$. Since the identity functor is exact it just boils down to noting that we are given $\mathcal{F}_3 \subseteq \mathcal{F}_1$ and that the injective objects are contained in $\mathcal{W} \cap \mathcal{F}_1 = \mathcal{F}_2$. \square

So reversing the direction of $(**)$ above, and using the above lemma we get the following diagram of functors on the level of homotopy categories.

$$\begin{array}{ccccc}
 \mathrm{Ho}(\mathcal{M}_2) & \xleftarrow{\mathrm{Lid}} & \mathrm{Ho}(\mathcal{M}_1) & \xleftarrow{\mathrm{Lid}} & \mathrm{Ho}(\mathcal{M}_1/\mathcal{M}_2) \\
 \uparrow & & \uparrow & & \uparrow \\
 & \xleftarrow{\mathrm{Rid}} & & \xleftarrow{\mathrm{Rid}} & \\
 \mathrm{Ho}(\mathcal{M}_1/\mathcal{M}_3) & \xleftarrow{\mathrm{Lid}} & \mathrm{Ho}(\mathcal{M}_1) & \xleftarrow{\mathrm{Lid}} & \mathrm{Ho}(\mathcal{M}_3) \\
 \downarrow & & \downarrow & & \downarrow \\
 & \xleftarrow{\mathrm{Rid}} & & \xleftarrow{\mathrm{Rid}} &
 \end{array}$$

(***)

This is a general form of Becker's butterfly diagram [Bec12, page 28]. We now make a few notes about the diagram before proceeding.

- We write all left adjoints on the top or on the left and we write all right adjoints on the bottom or right.
- The top row is a colocalization sequence.
- The bottom row is a localization sequence.
- The butterfly is pictured by identifying the two occurrences of $\mathrm{Ho}(\mathcal{M}_1)$ along the the middle vertical arrows. The middle vertical arrows are not literally the identity but these functors are canonical equivalences of $\mathrm{Ho}(\mathcal{M}_1)$. It is the identity however if you restrict to the full subcategory of cofibrant-fibrant objects.

Note that the condition $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ is all that is required for the butterfly diagram setup to exist. We now give our main construction theorem, Theorem 4.6, giving simple criteria for an induced recollement situation. The proof will make

use of the following proposition which relies on the uniqueness condition of Theorem 3.10. It can be useful on its own for spotting localizations of two injective model structures.

Proposition 4.5. *Let \mathcal{W} be a thick class in \mathcal{A} . Suppose $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ are injective model structures.*

- (1) *If $\mathcal{W} \cap \mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}$. Then $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$.*
- (2) *If $\mathcal{W}_2 \cap \mathcal{W} = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}$. Then $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$.*

Proof. First, note that $\mathcal{F}_2 \subseteq \mathcal{F}_1$ by the given. So taking left perps we get $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Also we are given that $\mathcal{W}_1 \subseteq \mathcal{W}$, and so $\mathcal{W}_1 \subseteq \mathcal{W}_2 \cap \mathcal{W}$ is automatic.

On the other hand, say $X \in \mathcal{W}_2 \cap \mathcal{W}$. Since $(\mathcal{W}_1, \mathcal{F}_1)$ is complete we can find a short exact sequence $0 \rightarrow B \rightarrow A \rightarrow X \rightarrow 0$ where $B \in \mathcal{F}_1$ and $A \in \mathcal{W}_1$. Since A and X are in the thick class \mathcal{W} we get that B is too. So $B \in \mathcal{F}_1 \cap \mathcal{W} = \mathcal{F}_2$. Since $(\mathcal{W}_2, \mathcal{F}_2)$ is a cotorsion pair the sequence $0 \rightarrow B \rightarrow A \rightarrow X \rightarrow 0$ must split, making X a direct summand of A . But then X must belong to \mathcal{W}_1 since the left side of a cotorsion pair is always closed under retracts. This shows $\mathcal{W}_1 = \mathcal{W}_2 \cap \mathcal{W}$ and it follows that $(\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$ is an injective Hovey triple and proves (1) because of the uniqueness part of Theorem 3.10. The proof of (2) is similar. The assumptions imply $\mathcal{F}_2 \subseteq \mathcal{W} \cap \mathcal{F}_1$. On the other hand, for $X \in \mathcal{F}_1 \cap \mathcal{W}$ use completeness of $(\mathcal{W}_2, \mathcal{F}_2)$ to find a s.e.s. $0 \rightarrow X \rightarrow F_2 \rightarrow W_2 \rightarrow 0$. But given that $F_2 \in \mathcal{W}$, thickness of \mathcal{W} implies $W_2 \in \mathcal{W} \cap \mathcal{W}_2 = \mathcal{W}_1$. So the sequence splits. \square

Theorem 4.6. *Let \mathcal{A} be a bicomplete abelian category with enough injectives and suppose we have three injective cotorsion pairs*

$$\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1), \quad \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2), \quad \mathcal{M}_3 = (\mathcal{W}_3, \mathcal{F}_3)$$

such that $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$. Then if either

- $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$,
- OR
- $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}_3$,

then $\mathcal{M}_1/\mathcal{M}_2$ is Quillen equivalent to \mathcal{M}_3 and $\mathcal{M}_1/\mathcal{M}_3$ is Quillen equivalent to \mathcal{M}_2 . In fact, there is a recollement

$$\begin{array}{ccccc} & & E(\mathcal{W}_2, \mathcal{F}_2) & & \\ & \swarrow & & \searrow & \\ \mathcal{F}_2/\sim & \xrightarrow{I} & \mathcal{F}_1/\sim & \xrightarrow{\lambda} & \mathcal{F}_3 \sim \\ & \nwarrow & & \swarrow & \\ & & C(\mathcal{W}_3, \mathcal{F}_3) & & \end{array}$$

$E(\mathcal{W}_3, \mathcal{F}_3)$ I

Here, the notation such as $E(\mathcal{W}_3, \mathcal{F}_3)$ means to take a special \mathcal{F}_3 -preenvelope by using enough injectives of the cotorsion pair $(\mathcal{W}_3, \mathcal{F}_3)$. Similarly the notation $C(\mathcal{W}_3, \mathcal{F}_3)$ means to take a special \mathcal{W}_3 -precover.

Proof. Consider again the butterfly diagram $(***)$ above consisting of all adjoint pairs. Assume hypothesis (i). That is, $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$. Since we also have $\mathcal{F}_3 \subseteq \mathcal{F}_1$ we also get $\mathcal{W}_1 \subseteq \mathcal{W}_3$. So applying part (i) of Proposition 4.5 (with \mathcal{W}_3 in place of \mathcal{W}) we conclude $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}_3, \mathcal{F}_1)$.

On the other hand, assume hypothesis (ii). Then applying part (ii) of Proposition 4.5 we again conclude $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}_3, \mathcal{F}_1)$.

So in either case, we have $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}_3, \mathcal{F}_1)$ and we proceed in the same fashion. First, note that $\mathcal{M}_1/\mathcal{M}_2$ and \mathcal{M}_3 have the same trivial objects. It follows then from Lemma 2.7 that the weak equivalences in $\mathcal{M}_1/\mathcal{M}_2$ coincide with those in \mathcal{M}_3 . So by definition, the identity adjunction $\mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_3$ is a Quillen equivalence. This means the vertical maps on the far right of (***) are equivalences. As we noted above the middle vertical maps in (***) are already canonical equivalences.

Second, note that the full subcategory of $\mathcal{M}_1/\mathcal{M}_3 = (\mathcal{W}_3, \mathcal{V}, \mathcal{F}_1)$ consisting of the cofibrant-fibrant subobjects is $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$. So we have $\text{Ho}(\mathcal{M}_1/\mathcal{M}_3) \cong (\mathcal{W}_3 \cap \mathcal{F}_1)/\sim = \mathcal{F}_2/\sim$ through the canonical equivalence. Thus on the level of the homotopy category associated to the full subcategory of cofibrant-fibrant subobjects, the diagram (***) becomes the following:

$$\begin{array}{ccccc}
 \mathcal{F}_2/\sim & \xleftarrow{E(\mathcal{W}_2, \mathcal{F}_2)} & \mathcal{F}_1/\sim & \xleftarrow{\text{Inclusion}} & (\mathcal{W}_2 \cap \mathcal{F}_1)/\sim \\
 \uparrow \text{id} & & \uparrow \text{id} & & \uparrow \text{id} \\
 \mathcal{F}_2/\sim & \xleftarrow{\text{Inclusion}} & \mathcal{F}_1/\sim & \xleftarrow{C(\mathcal{W}_2, \mathcal{F}_2)} & (\mathcal{W}_2 \cap \mathcal{F}_1)/\sim \\
 \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\
 \mathcal{F}_2/\sim & \xleftarrow{C(\mathcal{W}_3, \mathcal{F}_3)} & \mathcal{F}_1/\sim & \xleftarrow{E(\mathcal{W}_3, \mathcal{F}_3)} & \mathcal{F}_3/\sim \\
 \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\
 \mathcal{F}_2/\sim & \xleftarrow{\text{Inclusion}} & \mathcal{F}_1/\sim & \xleftarrow{\text{Inclusion}} & \mathcal{F}_3/\sim
 \end{array}$$

In particular, the left hand square shows that the inclusion map $I : \mathcal{F}_2/\sim \rightarrow \mathcal{F}_1/\sim$ has a left adjoint. So by [Kra06, Proposition 4.13.1] the localization sequence which makes up the bottom row induces the recollement.

□

We now record the projective version of the theorem.

Theorem 4.7. *Let \mathcal{A} be a bicomplete abelian category with enough projectives and suppose we have three projective cotorsion pairs*

$$\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1), \quad \mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2), \quad \mathcal{M}_3 = (\mathcal{C}_3, \mathcal{W}_3)$$

such that $\mathcal{C}_2, \mathcal{C}_3 \subseteq \mathcal{C}_1$. Then if either

- $\mathcal{W}_3 \cap \mathcal{C}_1 = \mathcal{C}_2$,
- OR
- $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{C}_2 \subseteq \mathcal{W}_3$,

then (the left localization) $\mathcal{M}_2 \setminus \mathcal{M}_1$ is Quillen equivalent to \mathcal{M}_3 and $\mathcal{M}_3 \setminus \mathcal{M}_1$ is Quillen equivalent to \mathcal{M}_2 . In fact, there is a recollement

$$\begin{array}{ccccc}
 & & E(\mathcal{C}_3, \mathcal{W}_3) & & \\
 & \swarrow & & \searrow & \\
 \mathcal{C}_2/\sim & \xrightarrow{I} & \mathcal{C}_1/\sim & \xrightarrow{I} & \mathcal{C}_3/\sim \\
 & \nwarrow & & \nwarrow & \\
 & & C(\mathcal{C}_2, \mathcal{W}_2) & & \\
 & & \rho & &
 \end{array}$$

Here, the notation such as $E(\mathcal{C}_3, \mathcal{W}_3)$ means to take a special \mathcal{W}_3 -preenvelope by using enough injectives of the cotorsion pair $(\mathcal{C}_3, \mathcal{W}_3)$. Similarly the notation $C(\mathcal{C}_3, \mathcal{W}_3)$ means to take a special \mathcal{C}_3 -precover.

Example 4.8. In terms of Theorem 4.6, Becker's method of recovering Krause's recollement boils down to the following observations. As in Example 3.12, let $dw\tilde{\mathcal{I}}$ be the class of all complexes of injective R -modules, $ex\tilde{\mathcal{I}}$ be the class of all exact complexes of injective R -modules and $dg\tilde{\mathcal{I}}$ be the class of all DG-injective complexes. Then we have the following injective cotorsion pairs:

$$\mathcal{M}_1 = (\mathcal{W}_1, dw\tilde{\mathcal{I}}), \quad \mathcal{M}_2 = (\mathcal{W}_2, ex\tilde{\mathcal{I}}), \quad \mathcal{M}_3 = (\mathcal{E}, dg\tilde{\mathcal{I}}).$$

It is easy to see that $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ and $\mathcal{E} \cap dw\tilde{\mathcal{I}} = ex\tilde{\mathcal{I}}$. So Theorem 4.6 gives a recollement which is Krause's recollement from [Kra05]. See also the introduction to Section 8.

We point out that this example readily extends to recover Krause's recollement for quasi-coherent sheaves over any scheme X . Indeed let (X, \mathcal{O}_X) be a scheme on a topological space X . Denote the category of all quasi-coherent sheaves of \mathcal{O}_X -modules by $\text{Qco}(X)$. We know $\text{Qco}(X)$ is a Grothendieck category, so it has enough injectives and a generator G . Therefore $\mathcal{A} = \text{Ch}(\text{Qco}(X))$ is also Grothendieck having $\{D^n(G)\}$ as a set of generators. Each of the model structures $\mathcal{M}_1, \mathcal{M}_2$ and \mathcal{M}_3 above exists in \mathcal{A} for the same formal reasons [BGH12]. We note that the set $\{D^n(G)\}$ of generators is in each of the classes $\mathcal{W}_1, \mathcal{W}_2$, and \mathcal{E} . This is because each of the classes $dw\tilde{\mathcal{I}}$, $ex\tilde{\mathcal{I}}$, and $dg\tilde{\mathcal{I}}$ consist of complexes of injectives I and so $\text{Ext}_{\mathcal{A}}^1(D^n(G), I) \cong \text{Ext}_{\text{Qco}(X)}^1(G, I_n) = 0$. The fact that the generators are in the left hand side of the cotorsion pairs is relevant. It is then automatic from [SS11] that each of $\mathcal{M}_1, \mathcal{M}_2$, and \mathcal{M}_3 are *small* in the sense of [Hov02] and so each corresponds to a cofibrantly generated model structure on \mathcal{A} .

5. THE GORENSTEIN INJECTIVE COTORSION PAIR

Let \mathcal{A} be any abelian category with enough injectives. It is clear that the canonical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ satisfies $\mathcal{I} \subseteq \mathcal{F}$ whenever $(\mathcal{W}, \mathcal{F})$ is another injective cotorsion pair. We now show two things. First, for any injective cotorsion pair $(\mathcal{W}, \mathcal{F})$ we have $\mathcal{F} \subseteq \mathcal{GI}$, where \mathcal{GI} is the class of Gorenstein injective objects. Second, whenever the class \mathcal{GI} forms the right half of a complete cotorsion pair $({}^\perp\mathcal{GI}, \mathcal{GI})$, then this is automatically an injective cotorsion pair too. These facts have an immediate application (Theorem 8.1) to a simple characterization of the Gorenstein injective chain complexes in $\text{Ch}(\mathcal{A})$ and leads us to consider a possible lattice structure on \mathcal{A} in Section 6. We start with the definition of a Gorenstein injective object in \mathcal{A} .

Definition 5.1. Let \mathcal{A} be an abelian category with enough injectives and let $M \in \mathcal{A}$. We call M *Gorenstein injective* if $M = Z_0 J$ for some exact complex J of injectives which remains exact after applying $\text{Hom}_{\mathcal{A}}(I, -)$ for any injective object I . We will also call such a complex J a *complete injective resolution* of M .

See [EJ00] for a basic reference on Gorenstein injective R -modules. It is shown in [BGH12] that $({}^\perp\mathcal{GI}, \mathcal{GI})$ is complete whenever R is any (left) Noetherian ring.

Theorem 5.2. Let \mathcal{A} be an abelian category with enough injectives and let \mathcal{GI} denote the class of Gorenstein injectives in \mathcal{A} . Then we have $\mathcal{F} \subseteq \mathcal{GI}$ whenever

$(\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair. Moreover, whenever $({}^\perp \mathcal{GI}, \mathcal{GI})$ is a complete cotorsion pair it is automatically an injective cotorsion pair too.

Proof. First, suppose $(\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair and let $F \in \mathcal{F}$. We must show that F is Gorenstein injective. We will construct a complete injective resolution of F . First, since \mathcal{A} has enough injectives we may take a usual injective coresolution of F . But this coresolution will actually remain exact after applying $\text{Hom}_{\mathcal{A}}(I, -)$ for any injective I since $(\mathcal{W}, \mathcal{F})$ is hereditary and since \mathcal{W} contains any injective I . So this gives us the right half $F \hookrightarrow J^*$ of a complete injective resolution of F .

To build the left half of a complete injective resolution use that $(\mathcal{W}, \mathcal{F})$ has enough injectives to get a short exact sequence

$$(*) \quad 0 \longrightarrow F' \longrightarrow J \longrightarrow F \longrightarrow 0,$$

where $J \in \mathcal{W}$ and $F' \in \mathcal{F}$. As \mathcal{F} is closed under extensions, and since $F, F' \in \mathcal{F}$, it follows that $J \in \mathcal{W} \cap \mathcal{F}$. That is, J must be injective. Since $F' \in \mathcal{F}$ we again have $\text{Ext}_{\mathcal{A}}^1(I, F') = 0$ for all injectives I . Hence the exact sequence $(*)$ stays exact under application of the functor $\text{Hom}_{\mathcal{A}}(I, -)$ for every whenever I is injective. Iterating this process allows us to construct the left half of a complete injective resolution $J_* \twoheadrightarrow F$. We compose to set $J = J_* \twoheadrightarrow F \hookrightarrow J^*$ with $M = Z_0 J$ making J the desired complete injective resolution of M .

Next, set $\mathcal{V} = {}^\perp \mathcal{GI}$ and suppose that $(\mathcal{V}, \mathcal{GI})$ is a complete cotorsion pair. If M is Gorenstein injective then it follows from the definition that $\text{Ext}_{\mathcal{A}}^n(I, M) = 0$ for any injective object I and $n \geq 1$. So \mathcal{V} contains the injective objects. Next, we claim that the class \mathcal{GI} is cosyzygy closed. Indeed it is clear that if M is Gorenstein injective with complete injective resolution J , then $Z_n J$ are also all Gorenstein injective by definition. In particular we have the short exact sequence $0 \rightarrow M \rightarrow J_0 \rightarrow Z_{-1} J \rightarrow 0$ with $Z_{-1} J$ also Gorenstein injective. So if $0 \rightarrow M \rightarrow I \rightarrow Z \rightarrow 0$ is any other short exact sequence with I injective we get $\text{Ext}_{\mathcal{A}}^1(V, Z) \cong \text{Ext}_{\mathcal{A}}^2(V, M) \cong \text{Ext}_{\mathcal{A}}^1(V, Z_{-1} J) = 0$. So the class \mathcal{GI} is cosyzygy closed. Since \mathcal{A} has enough injectives we conclude $(\mathcal{V}, \mathcal{GI})$ is hereditary from Lemma 2.3. Now $(\mathcal{V}, \mathcal{GI})$ satisfies the hypotheses of Lemma 3.4 and so \mathcal{V} is thick. Finally, since we have shown that \mathcal{V} is thick and contains the injectives we get from part (2) of Theorem 3.5 that $(\mathcal{V}, \mathcal{GI})$ is an injective cotorsion pair. \square

5.1. The Gorenstein projective cotorsion pair. We now state the dual result concerning the Gorenstein projectives.

Definition 5.3. Let \mathcal{A} be an abelian category with enough projectives and let $M \in \mathcal{A}$. We call M *Gorenstein projective* if $M = Z_0 Q$ for some exact complex Q of projectives which remains exact after applying $\text{Hom}_{\mathcal{A}}(-, P)$ for any projective object P . We will also call such a complex Q a *complete projective resolution* of M .

Again, see [EJ00] for a basic reference on Gorenstein projective R -modules. It is shown in [BGH12] that $(\mathcal{GP}, \mathcal{GP}^\perp)$ is in fact cogenerated by a set, so complete, whenever R is a (left) coherent ring in which all flat modules have finite projective dimension. We pause now to comment on the extraordinarily large class of rings satisfying the condition that all flat modules have finite projective dimension.

In [Sim74], Simson gives a short proof of the following: If a ring R has cardinality at most \aleph_n then the maximal projective dimension of a flat module is at most $n+1$. This amazing result doesn't depend on whether the ring is commutative or Noetherian or anything and so virtually every ring we see in practice has the property that all flat modules have finite projective dimension. On the other hand, putting cardinality aside, Enochs, Jenda and López-Ramos have considered rings in [EJLR] they call n -perfect. These are rings in which all flat modules have projective dimension at most n . In that paper and in other papers coauthored by Enochs they give numerous examples of n -perfect rings. In particular, a perfect ring is 0-perfect and any n -Gorenstein ring is n -perfect. In this language the above result of Simson says that any ring R with cardinality at most \aleph_n is $(n+1)$ -perfect. There is more. A main example of an n -perfect ring is a commutative Noetherian ring of finite Krull dimension n . This follows from [Jen70] and [GR71]. Finally, we point out a generalization of this due to Peter Jørgensen. In the article [Jør05], he shows that every flat module has finite projective dimension whenever R is right-Noetherian and has a dualizing complex.

We note that our condition of saying R has finite projective dimension for each flat module is more general than saying R is n -perfect because we are not assuming an upper bound on the projective dimensions.

Theorem 5.4. *Let \mathcal{A} be an abelian category with enough projectives and let \mathcal{GP} denote the class of Gorenstein projectives in \mathcal{A} . Then we have $\mathcal{C} \subseteq \mathcal{GP}$ whenever $(\mathcal{C}, \mathcal{W})$ is a projective cotorsion pair. Moreover, whenever $(\mathcal{GP}, \mathcal{GP}^\perp)$ is a complete cotorsion pair it is automatically a projective cotorsion pair too.*

6. THE SEMILATTICE OF INJECTIVE COTORSION PAIRS IN $R\text{-MOD}$ AND $\text{Ch}(R)$

Let R be a ring. For the remainder of the paper we assume that \mathcal{A} is either the category of R -modules or the category of chain complexes of R -modules. That is, now \mathcal{A} denotes either $R\text{-Mod}$ or $\text{Ch}(R)$. The theory for rings serves as a guide for what might be true in more general settings. We point the reader in particular to the important work in [SŠ11] and [Što11] on complete cotorsion pairs in Grothendieck categories. Most of the results ahead in the current paper should be obtainable for a general Grothendieck category \mathcal{A} as long as one is willing to assume there is a generator for \mathcal{A} in the left side of any injective cotorsion pair we are interested in. Having said this, in light of Section 5 and Theorem 4.6, it makes sense to consider whether or not the injective cotorsion pairs form a lattice. We now make a brief investigation, again for \mathcal{A} being $R\text{-Mod}$ or $\text{Ch}(R)$. Ultimately we are only able to show that arbitrary suprema of injective cotorsion pairs exist and so we call the ordering a *semilattice*. The question remains open as to whether or not infima of injective cotorsion pairs are again injective cotorsion pairs.

Lemma 6.1. *Suppose $\{(\mathcal{C}_i, \mathcal{D}_i)\}_{i \in I}$ is a collection of cotorsion pairs in \mathcal{A} , each cogenerated by some class \mathcal{S}_i and generated by some class \mathcal{T}_i .*

- (1) *$(\perp(\cap_{i \in I} \mathcal{D}_i), \cap_{i \in I} \mathcal{D}_i)$ is a cotorsion pair cogenerated by the class $\cup_{i \in I} \mathcal{S}_i$. In particular, if each \mathcal{S}_i is a set, then the cotorsion pair is also cogenerated by a set.*
- (2) *$(\cap_{i \in I} \mathcal{C}_i, (\cap_{i \in I} \mathcal{C}_i)^\perp)$ is a cotorsion pair generated by the class $\cup_{i \in I} \mathcal{T}_i$. In particular, if each \mathcal{T}_i is a set, then the cotorsion pair is also generated by a set.*

Proof. It is straightforward to check that $(\cup_{i \in I} \mathcal{S}_i)^\perp = \cap_{i \in I} \mathcal{D}_i$ which proves (1) and ${}^\perp(\cup_{i \in I} \mathcal{T}_i) = \cap_{i \in I} \mathcal{C}_i$ which proves (2). \square

We can put a partial order on the class of all cotorsion pairs using containment of either the classes on the left side or the classes on the right side. For the theory of model categories it is best to change the ordering depending on whether we are focusing on injective or projective cotorsion pairs. But following either ordering, Lemma 6.1 guarantees suprema and infima so that a partial ordering gives a complete lattice on the class of all cotorsion pairs. It is nontrivial that restricting the partial ordering to just those cotorsion pairs which are cogenerated by a set forms a complete sublattice. But this follows from the fact that $\cap_{i \in I} \mathcal{C}_i$ is *deconstructible* (as defined in [Što11]) whenever each $(\mathcal{C}_i, \mathcal{D}_i)$ is cogenerated by a set [Što11, Proposition 2.9].

6.1. The semilattice of injective cotorsion pairs. Suppose we have two injective cotorsion pairs $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ in \mathcal{A} . Then we define $\mathcal{M}_2 \preceq_r \mathcal{M}_1$ if and only if $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Note that this happens if and only if the inclusion functor $\mathcal{M}_2 \rightarrow \mathcal{M}_1$ is a right Quillen functor. With respect to \preceq_r we have that the canonical injective cotorsion pair is the least element with respect to this ordering and from Theorem 5.2 the Gorenstein injective cotorsion pair, whenever it exists, is the maximum element. We know it exists whenever R is Noetherian by [BGH12].

Proposition 6.2. *Let $\{(\mathcal{W}_i, \mathcal{F}_i)\}_{i \in I}$ be a collection of injective cotorsion pairs each cogenerated by a set \mathcal{S}_i . Then its supremum $\bigvee_{i \in I} (\mathcal{W}_i, \mathcal{F}_i) = (\cap_{i \in I} \mathcal{W}_i, (\cap_{i \in I} \mathcal{W}_i)^\perp)$ is also an injective cotorsion pair cogenerated by a set.*

Proof. Since each $(\mathcal{W}_i, \mathcal{F}_i)$ is cogenerated by a set, each class \mathcal{W}_i is deconstructible by [Što11]. Then by [Što11, Proposition 2.9 (2)] it follows that $\cap_{i \in I} \mathcal{W}_i$ is also deconstructible. This implies the cotorsion pair $(\cap_{i \in I} \mathcal{W}_i, (\cap_{i \in I} \mathcal{W}_i)^\perp)$ is cogenerated by a set, and so is complete. Since each \mathcal{W}_i is thick and contains the injective objects we get that $\cap_{i \in I} \mathcal{W}_i$ is also thick and contains the injectives. So $(\cap_{i \in I} \mathcal{W}_i, (\cap_{i \in I} \mathcal{W}_i)^\perp)$ is an injective cotorsion pair by Theorem 3.5. \square

Remark 6. Proposition 6.2 says that the ordering on the class of all injective cotorsion pairs that are cogenerated by a set is a “complete join-semilattice” sitting inside the complete lattice of all cotorsion pairs that are cogenerated by a set. By a complete join-semilattice we mean that the supremum of any given set exists. The author doesn’t know whether or not infima exist. That is, let $\{(\mathcal{W}_i, \mathcal{F}_i)\}_{i \in I}$ be a collection of injective cotorsion pairs each cogenerated by a set \mathcal{S}_i . Then we know from Lemma 6.1 that $({}^\perp(\cap_{i \in I} \mathcal{F}_i), \cap_{i \in I} \mathcal{F}_i)$ is the infimum in the lattice of all cotorsion pairs that are cogenerated by a set. We would like to know if this cotorsion pair is injective. (The only thing not clear is whether or not the left class is closed under taking cokernels of monomorphisms between its objects. Equivalently, does $[{}^\perp(\cap_{i \in I} \mathcal{F}_i)] \cap [\cap_{i \in I} \mathcal{F}_i]$ consist only of injective modules?)

6.2. The semilattice of projective cotorsion pairs. On the other hand we have the semilattice of projective cotorsion pairs in \mathcal{A} . We give the simple proof of the dual result below. It is necessarily different due to the fact that we must always *cogenerate* a cotorsion pair by a set to get completeness.

Suppose that $\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1)$ and $\mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2)$ are projective cotorsion pairs in \mathcal{A} . Then we define $\mathcal{M}_2 \preceq_l \mathcal{M}_1$ if and only if $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Note that this happens if and only if the inclusion functor $\mathcal{M}_2 \rightarrow \mathcal{M}_1$ is a left Quillen functor. With respect to \preceq_l we have that the canonical projective cotorsion pair is the least element with respect to the ordering and from Theorem 5.4 the Gorenstein projective cotorsion pair, whenever it exists, is the maximum element. We know it exists whenever R is a coherent ring in which all flat modules have finite projective dimension by [BGH12].

Proposition 6.3. *Let $\{(\mathcal{C}_i, \mathcal{W}_i)\}_{i \in I}$ be a collection of projective cotorsion pairs each cogenerated by a set \mathcal{S}_i . Then its supremum, which is the cotorsion pair $\bigvee_{i \in I} (\mathcal{C}_i, \mathcal{W}_i) = (\bigcap_{i \in I} \mathcal{C}_i, \bigcap_{i \in I} \mathcal{W}_i)$, is also a projective cotorsion pair cogenerated by a set.*

Proof. Here it is clear that $\bigcap_{i \in I} \mathcal{W}_i$ is thick and contains the projectives since each \mathcal{W}_i does. Also, if $\{\mathcal{S}_i\}_{i \in I}$ represents the cogenerating sets, then $\bigcup_{i \in I} \mathcal{S}_i$ is a cogenerating set by Lemma 6.1. So $\bigvee_{i \in I} (\mathcal{C}_i, \mathcal{W}_i)$ is complete and so a projective cotorsion pair by Theorem 3.6. \square

6.3. Examples. In general, the semilattice of injective (or projective) cotorsion pairs on $\text{Ch}(R)$ is always more interesting than the one on $R\text{-Mod}$ and depends much on the global (Gorenstein) dimension of R . We will look at examples concerning $\text{Ch}(R)$ at the end of Section 7.

For a generic ring R , there seems to be just two injective cotorsion pairs on $R\text{-Mod}$ that are typically of interest. The first is the canonical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ and the second the Gorenstein injective cotorsion pair $(\mathcal{W}, \mathcal{GI})$ when R is Noetherian. (There is in fact an analog that exists whenever R is only coherent. See Vista 8.2. Also see the introduction to Section 8 for more on the existence of the Gorenstein injective cotorsion pair.) But a ring can certainly have more than these two injective cotorsion pairs. For example, if R is Noetherian then as we see in the next section there are several injective cotorsion pairs on $\text{Ch}(R)$. But $\text{Ch}(R)$ is really just a graded version of the ring $R[x]/(x^2)$, and $R[x]/(x^2)\text{-Mod}$ will have these analogous injective cotorsion pairs as well.

We note the following simplification for a ring of finite global dimension.

Example 6.4. Suppose R is a ring with $\text{gl.dim}(R) < \infty$. Then the Gorenstein injective modules coincide with the injective modules. So in this case there is only one injective cotorsion pair in $R\text{-Mod}$, the categorical one. Similarly there is only one projective cotorsion pair in $R\text{-Mod}$, as the canonical projective cotorsion pair coincides with the Gorenstein projective cotorsion pair.

7. LIFTING FROM MODULES TO CHAIN COMPLEXES

We now prove a general theorem which allows one to obtain several injective (resp. projective) model structures on $\text{Ch}(R)$ by starting with a single injective (resp. projective) model structure on $R\text{-Mod}$. We use the following notations which were introduced in [Gil04] and [Gil08].

Definition 7.1. Given a class of R -modules \mathcal{C} , we define the following classes of chain complexes in $\text{Ch}(R)$.

- (1) $dw\tilde{\mathcal{C}}$ is the class of all chain complexes with $C_n \in \mathcal{C}$.
- (2) $ex\tilde{\mathcal{C}}$ is the class of all exact chain complexes with $C_n \in \mathcal{C}$.

(3) $\widetilde{\mathcal{C}}$ is the class of all exact chain complexes with cycles $Z_n C \in \mathcal{C}$.

The “dw” is meant to stand for “degreewise” while the “ex” is meant to stand for “exact”. When \mathcal{C} is the class of projective (resp. injective, resp. flat) modules, then $\widetilde{\mathcal{C}}$ are the categorical projective (resp. injective, resp. flat) chain complexes.

Moreover, if we are given any cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $R\text{-Mod}$, then following [Gil04] we will denote $\widetilde{\mathcal{F}}^\perp$ by $dg\widetilde{\mathcal{C}}$ and ${}^\perp\widetilde{\mathcal{C}}$ by $dg\widetilde{\mathcal{F}}$.

7.1. Injective cotorsion pairs in $\text{Ch}(R)$ from one in $R\text{-Mod}$. Let R be any ring and $(\mathcal{W}, \mathcal{F})$ be an injective cotorsion pair in $R\text{-Mod}$ which is cogenerated by a set. We now show how this information allows for the construction of six injective cotorsion pairs, also each cogenerated by a set, in $\text{Ch}(R)$. However, depending on the pair $(\mathcal{W}, \mathcal{F})$ that we start with and the particular ring R , some of these six will coincide. See the examples ahead in Section 7.3.

Theorem 7.2. *Let $(\mathcal{W}, \mathcal{F})$ be an injective cotorsion pair of R -modules cogenerated by some set. Then the following are also each injective cotorsion pairs in $\text{Ch}(R)$, and cogenerated by sets.*

- (1) $({}^\perp dw\widetilde{\mathcal{F}}, dw\widetilde{\mathcal{F}})$
- (2) $({}^\perp ex\widetilde{\mathcal{F}}, ex\widetilde{\mathcal{F}})$
- (3) $(dg\widetilde{\mathcal{W}}, \widetilde{\mathcal{F}})$
- (4) $(dw\widetilde{\mathcal{W}}, (dw\widetilde{\mathcal{W}})^\perp)$
- (5) $(ex\widetilde{\mathcal{W}}, (ex\widetilde{\mathcal{W}})^\perp)$
- (6) $(\widetilde{\mathcal{W}}, dg\widetilde{\mathcal{F}})$

Proof. This time we give the proofs for the projective case and this appears in Theorem 7.3 below. We note that the only difference between the projective and injective case is that we are always *cogenerating* by a set to get completeness of the cotorsion pair. For (1)–(3) one can cogenerate by a set due to [Gil08, Propositions 4.3, 4.4, and 4.6] and for (4)–(6) one can use the theory of deconstructible classes in [Štö11]. We also point out that using the proper language this can be extended to Grothendieck categories that have a generator in \mathcal{W} , rather than just $R\text{-Mod}$, because of the generality in both of the above papers. \square

7.2. Projective cotorsion pairs in $\text{Ch}(R)$ from one in $R\text{-Mod}$. Let R be any ring and $(\mathcal{C}, \mathcal{W})$ be a projective cotorsion pair in $R\text{-Mod}$ which is cogenerated by a set. We have the dual statement to Theorem 7.2 which we now prove.

Theorem 7.3. *Let $(\mathcal{C}, \mathcal{W})$ be a projective cotorsion pair of R -modules cogenerated by some set. Then the following are also each projective cotorsion pairs in $\text{Ch}(R)$, and cogenerated by sets.*

- (1) $(dw\widetilde{\mathcal{C}}, (dw\widetilde{\mathcal{C}})^\perp)$
- (2) $(ex\widetilde{\mathcal{C}}, (ex\widetilde{\mathcal{C}})^\perp)$
- (3) $(\widetilde{\mathcal{C}}, dg\widetilde{\mathcal{W}})$
- (4) $({}^\perp dw\widetilde{\mathcal{W}}, dw\widetilde{\mathcal{W}})$
- (5) $({}^\perp ex\widetilde{\mathcal{W}}, ex\widetilde{\mathcal{W}})$
- (6) $(dg\widetilde{\mathcal{C}}, \widetilde{\mathcal{W}})$

Proof. The proofs for (1)–(3) are all similar as are the proofs for (4)–(6).

Lets prove (6). First, it follows from [Gil04, Section 3] that $({}^\perp \widetilde{\mathcal{W}}, \widetilde{\mathcal{W}})$ is a cotorsion pair. It is easy to show that $\widetilde{\mathcal{W}}$ is thick because \mathcal{W} is thick. (In particular, note that any short exact sequence $0 \rightarrow W' \rightarrow W \rightarrow W'' \rightarrow 0$ of complexes in $\widetilde{\mathcal{W}}$ gives rise to a short exact sequence $0 \rightarrow Z_n W' \rightarrow Z_n W \rightarrow Z_n W'' \rightarrow 0$ on the level of cycles because the complex W' is exact.) Next, recall that a projective chain complex is one which is exact with projective cycles. Since \mathcal{W} contains the projective modules, we get that $\widetilde{\mathcal{W}}$ contains the projective complexes. So since $\widetilde{\mathcal{W}}$ is thick and contains the projectives it will follow from Theorem 3.6 that $({}^\perp \widetilde{\mathcal{W}}, \widetilde{\mathcal{W}})$ is a projective cotorsion pair once we know it is complete. Note here that if we had started with $({}^\perp dw\widetilde{\mathcal{W}}, dw\widetilde{\mathcal{W}})$ or $({}^\perp ex\widetilde{\mathcal{W}}, ex\widetilde{\mathcal{W}})$ then again we get that each of these is a cotorsion pair by [Gil08, Propositions 3.2 and 3.3] and similarly we argue that both $dw\widetilde{\mathcal{W}}$ and $ex\widetilde{\mathcal{W}}$ are thick and contain the projectives. But finally, it was shown in [Gil08, Propositions 4.3–4.6] that each of the cotorsion pairs $({}^\perp dw\widetilde{\mathcal{W}}, dw\widetilde{\mathcal{W}})$, $({}^\perp ex\widetilde{\mathcal{W}}, ex\widetilde{\mathcal{W}})$ and $({}^\perp \widetilde{\mathcal{W}}, \widetilde{\mathcal{W}})$ are cogenerated by a set. This proves (4)–(6).

Now lets prove (1)–(3). It follows again from [Gil04] and [Gil08] that these are all cotorsion pairs. Lets focus on (2) for example, since (1) and (3) will be similar. First, from [Gil08, Proposition 3.3] we note that for this cotorsion pair $(ex\widetilde{\mathcal{C}}, (ex\widetilde{\mathcal{C}})^\perp)$ the right class $(ex\widetilde{\mathcal{C}})^\perp$ equals the class of all complexes W for which $W_n \in \mathcal{W}$ and such that $Hom(C, W)$ is exact whenever $C \in ex\widetilde{\mathcal{C}}$. (Hom is defined in Section 2. By Lemma 2.1 we get that $Hom(C, W)$ is exact if and only if any chain map $f : \Sigma^n C \rightarrow W$ is null homotopic if and only if any chain map $f : C \rightarrow \Sigma^n W$ is null homotopic.) In light of Theorem 3.6 we wish to show that this class $(ex\widetilde{\mathcal{C}})^\perp$ is thick and contains the projectives and that the cotorsion pair $(ex\widetilde{\mathcal{C}}, (ex\widetilde{\mathcal{C}})^\perp)$ is complete. The projective complexes are easily seen to be in $(ex\widetilde{\mathcal{C}})^\perp$ by its above description (recall \mathcal{W} contains the projectives and any chain map into a projective is null). Next, $(ex\widetilde{\mathcal{C}})^\perp$ is clearly closed under retracts since it is the right side of a cotorsion pair. To complete the thickness claim suppose that $0 \rightarrow W' \rightarrow W \rightarrow W'' \rightarrow 0$ is a short exact sequence of complexes. If any two of the three W', W, W'' are in $(ex\widetilde{\mathcal{C}})^\perp$ then note that since \mathcal{W} is thick we get that all of the W'_n, W_n, W''_n are in \mathcal{W} . It now follows that for any $C \in ex\widetilde{\mathcal{C}}$ we will always get a short exact sequence of Hom -complexes $0 \rightarrow Hom(C, W') \rightarrow Hom(C, W) \rightarrow Hom(C, W'') \rightarrow 0$ (because Ext vanishes degreeewise). So now if any two of the three complexes $Hom(C, W'), Hom(C, W), Hom(C, W'')$ are exact then the third is automatically exact due to the long exact sequence in homology. This completes the proof that $(ex\widetilde{\mathcal{C}})^\perp$ is thick. Finally it is left to show that $(ex\widetilde{\mathcal{C}}, (ex\widetilde{\mathcal{C}})^\perp)$ is complete. We use the results in [Što11] pointing out that a class which is the left side of a cotorsion pair is *deconstructible* if and only if that cotorsion pair is cogenerated by set. It follows from [Što11, Theorem 4.2] that $dw\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ are deconstructible since \mathcal{C} is. So the only cotorsion pair left is $(ex\widetilde{\mathcal{C}}, (ex\widetilde{\mathcal{C}})^\perp)$. But here note that $ex\widetilde{\mathcal{C}} = dw\widetilde{\mathcal{C}} \cap \mathcal{E}$ where \mathcal{E} is the class of exact complexes. Since \mathcal{E} is the left side of a cotorsion pair cogenerated by a set it is deconstructible and since $dw\widetilde{\mathcal{C}}$ is also deconstructible, it follows from [Što11, Proposition 2.9] that $ex\widetilde{\mathcal{C}}$ is also deconstructible. \square

7.3. Examples and homological dimensions. We briefly discussed the semi-lattices of injective and projective cotorsion pairs on $R\text{-Mod}$ in Section 6.3. We continue that discussion now by looking in more detail at the basic examples of

model structures on $\text{Ch}(R)$ induced from the categorical injective (resp. projective) cotorsion pair and the Gorenstein injective (resp. Gorenstein projective) cotorsion pair. The basic theme is that for a Noetherian ring R , the semilattice for $\text{Ch}(R)$ becomes more complicated as we move from R having finite global dimension, to R being Gorenstein, to a general Noetherian R . In the next section we look in more detail at the models induced by the Gorenstein injective and Gorenstein projective pairs.

We will give projective examples here and leave the obvious dual statements concerning injective cotorsion pairs to the reader. Let \mathcal{P} denote the class of projective modules and $(\mathcal{P}, \mathcal{A})$ the canonical projective cotorsion pair. Consider the six projective cotorsion pairs on $\text{Ch}(R)$ induced by $(\mathcal{P}, \mathcal{A})$ using Theorem 7.3. We see that they are in fact only four distinct pairs. They are $(dw\tilde{\mathcal{P}}, (dw\tilde{\mathcal{P}})^\perp)$, $(ex\tilde{\mathcal{P}}, (ex\tilde{\mathcal{P}})^\perp)$, $(\tilde{\mathcal{P}}, \text{Ch}(R))$, and $(dg\tilde{\mathcal{P}}, \mathcal{E})$. We have the following observation.

Proposition 7.4. *Let R be any ring with $\text{gl.dim}(R) < \infty$. Then we have $ex\tilde{\mathcal{P}} = \tilde{\mathcal{P}}$ and $dw\tilde{\mathcal{P}} = dg\tilde{\mathcal{P}}$.*

Proof. $dg\tilde{\mathcal{P}} \subseteq dw\tilde{\mathcal{P}}$ is automatic. To show $dw\tilde{\mathcal{P}} \subseteq dg\tilde{\mathcal{P}}$ let $P \in dw\tilde{\mathcal{P}}$ and let $E \in \mathcal{E}$. We must show $\text{Ext}^1(P, E) = 0$. Since $\text{gl.dim}(R) = d < \infty$, it is easy to argue that E has finite projective dimension in $\text{Ch}(R)$ [because any d th syzygy must be exact and inherit projective cycles.] Letting

$$0 \rightarrow P^d \rightarrow P^{d-1} \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow E \rightarrow 0$$

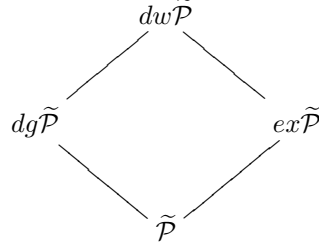
be a finite projective resolution we conclude by dimension shifting that $\text{Ext}^1(P, E) = \text{Ext}^{d+1}(P, P^d)$. But $\text{Ext}^{d+1}(P, P^d) = 0$ because $(dw\tilde{\mathcal{P}}, (dw\tilde{\mathcal{P}})^\perp)$ is a projective cotorsion pair. Thus $dw\tilde{\mathcal{P}} = dg\tilde{\mathcal{P}}$. Also, $ex\tilde{\mathcal{P}} = \tilde{\mathcal{P}}$ because given any $P \in ex\tilde{\mathcal{P}}$ and an R -module N , we can dimension shift $\text{Ext}^1(Z_n P, N) = \text{Ext}^{d+1}(Z_{n-d} P, N) = 0$. \square

So the next two examples clarify further the models obtained on $\text{Ch}(R)$ from $(\mathcal{P}, \mathcal{A})$ using Theorem 7.3.

Example 7.5. Say R has finite global dimension. Then we saw in Example 6.4 that $(\mathcal{P}, \mathcal{A}) = (\mathcal{GP}, \mathcal{W})$ is the only projective cotorsion pair on $R\text{-Mod}$. Proposition 7.4 tells us that we only get two distinct projective cotorsion pairs coming from Theorem 7.3 in this case. This first is $(dw\tilde{\mathcal{P}}, \mathcal{E})$ which is the projective model for the derived category of R and the other is the trivial projective model structure $(\tilde{\mathcal{P}}, \mathcal{A})$. It follows from Theorem 8.3 that $(dw\tilde{\mathcal{P}}, \mathcal{E})$ is actually the Gorenstein projective pair in $\text{Ch}(R)$, sitting on top of the semilattice while $(\tilde{\mathcal{P}}, \mathcal{A})$ is the canonical projective pair sitting at the bottom of the semilattice. The author doesn't know whether or not there are others on $\text{Ch}(R)$ besides these two when $\text{gl.dim}(R) < \infty$.

Example 7.6. Now suppose that R has infinite global dimension. Then we have all four generally distinct pairs $(dw\tilde{\mathcal{P}}, (dw\tilde{\mathcal{P}})^\perp)$, $(ex\tilde{\mathcal{P}}, (ex\tilde{\mathcal{P}})^\perp)$, $(\tilde{\mathcal{P}}, \text{Ch}(R))$, and $(dg\tilde{\mathcal{P}}, \mathcal{E})$. We note $\tilde{\mathcal{P}}$ is the class of categorical projective complexes and so $(\tilde{\mathcal{P}}, \tilde{\mathcal{P}}^\perp)$ is trivial as a model structure, and not of interest. Next, $dg\tilde{\mathcal{P}}$ is the class of DG-projective complexes and $(dg\tilde{\mathcal{P}}, \mathcal{E})$ is the usual *projective model structure* on $\text{Ch}(R)$ having homotopy category the usual derived category $\mathcal{D}(R)$. The model structure associated to $(dw\tilde{\mathcal{P}}, (dw\tilde{\mathcal{P}})^\perp)$ appears in [BGH12] where it is called the *Proj model structure* on $\text{Ch}(R)$. This model structure has also appeared in [Pos11] where its homotopy category was called the *contraderived category* of R . The model

structure $(ex\tilde{\mathcal{P}}, (ex\tilde{\mathcal{P}})^\perp)$ also appears in [BGH12] where it is called the *exact Proj model structure* on $\text{Ch}(R)$. Its homotopy category is the (projective) stable derived category $S(R)$ introduced in [BGH12]. For a general ring R we have the portion of the semilattice shown below.



We point out that there are two more model structures on $\text{Ch}(R)$ distinct from the above that exist whenever R is a coherent ring in which all flat modules have finite projective dimension. These will appear in [BGH12].

Having considered projective models on $\text{Ch}(R)$ induced from the canonical projective pair $(\mathcal{P}, \mathcal{A})$ via Theorem 7.3, we now turn to models induced from the Gorenstein projective cotorsion pair $(\mathcal{GP}, \mathcal{W})$. More on this appears in the next Section 8. We now just look at the case when R is a Gorenstein ring.

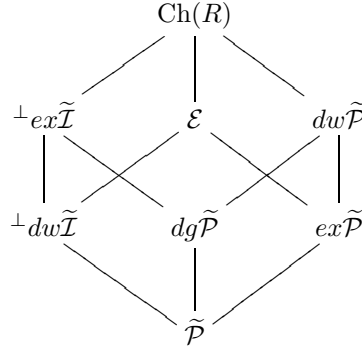
Example 7.7. Let R be a Gorenstein ring and again assume $\text{gl.dim}(R) = \infty$. Recall that R must have finite injective dimension when considered as a module over itself and that these dimensions must coincide. Here assume $\text{id}(R) = d$. Also recall that a module M over R has finite injective dimension if and only if it has finite projective dimension if and only if it has finite flat dimension and that if this is the case then all these dimensions must be $\leq d$. Call these the modules of *finite R -dimension* and let \mathcal{W} denote the class of all these modules. Then it was shown in [Hov02] that $(\mathcal{W}, \mathcal{GI})$ is a complete cotorsion pair where \mathcal{GI} are the Gorenstein injective complexes and that $(\mathcal{GP}, \mathcal{W})$ is a complete cotorsion pair where \mathcal{GP} are the Gorenstein projective complexes. Then we have the following Gorenstein version of Proposition 7.4

Proposition 7.8. *Let R be a Gorenstein ring with $\text{id}(R) = d$ and let $(\mathcal{GP}, \mathcal{W})$ be the Gorenstein projective cotorsion pair. Then $dw\tilde{\mathcal{GP}} = dg\tilde{\mathcal{GP}}$ and $ex\tilde{\mathcal{GP}} = \tilde{\mathcal{GP}}$.*

Proof. Let $G \in ex\tilde{\mathcal{GP}}$. We wish to show $Z_n G \in \mathcal{GP}$. So let $W \in \mathcal{W}$ and we must show $\text{Ext}_R^1(Z_n G, W) = 0$. But since $\text{Ext}_R^i(C, W) = 0$ for all $C \in \mathcal{GP}$ we can dimension shift to get $\text{Ext}_R^1(Z_n G, W) \cong \text{Ext}_R^{d+1}(Z_{n-d} G, W)$. But since W must have finite injective dimension less than or equal to d we get that this last group equals 0. Therefore $G \in \tilde{\mathcal{GP}}$. So $ex\tilde{\mathcal{GP}} = \tilde{\mathcal{GP}}$. The fact that $dw\tilde{\mathcal{GP}} = dg\tilde{\mathcal{GP}}$ is true by combining [GH10, Theorem 3.11] with Theorem 8.1 below. \square

We conclude that when R is Gorenstein of infinite global dimension, applying Theorem 7.3 to both the categorical projective and the Gorenstein projective pairs generally leads to 8 model structures on $\text{Ch}(R)$. This is illustrated concretely in the next example where all 8 model structures are distinct.

Example 7.9. Let $R = \mathbb{Z}_4$, the ring of integers mod 4 and consider $\text{Ch}(R)$. Then as described in Example 7.6, the projective cotorsion pair $(\mathcal{P}, \mathcal{A})$ on $R\text{-Mod}$ gives rise to the four projective cotorsion pairs $(dw\tilde{\mathcal{P}}, (dw\tilde{\mathcal{P}})^\perp)$, $(ex\tilde{\mathcal{P}}, (ex\tilde{\mathcal{P}})^\perp)$, $(\tilde{\mathcal{P}}, \text{Ch}(R))$, and $(dg\tilde{\mathcal{P}}, \mathcal{E})$ in $\text{Ch}(R)$. These classes are indeed distinct because the complex $\cdots \mathbb{Z}/4 \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{\times 2} \mathbb{Z}/4 \cdots$ is in $ex\tilde{\mathcal{P}}$ but not $\tilde{\mathcal{P}}$, and so also $dw\tilde{\mathcal{P}}$ but not $dg\tilde{\mathcal{P}}$. Recall that R is quasi-Frobenius meaning that the class of injective modules coincides with the class of projective modules. It follows that the Gorenstein projective cotorsion pair on $R\text{-Mod}$ is $(\mathcal{A}, \mathcal{I})$ where \mathcal{I} is the class of injective/projective R -modules. The four associated projective cotorsion pairs from Example 7.7 turn out to be $(\text{Ch}(R), \tilde{\mathcal{I}})$ and $(\mathcal{E}, dg\tilde{\mathcal{I}})$ and $({}^\perp ex\tilde{\mathcal{I}}, ex\tilde{\mathcal{I}})$ and $({}^\perp dw\tilde{\mathcal{I}}, dw\tilde{\mathcal{I}})$. These eight classes of cofibrant objects are distinct and are related to each other as shown in the cube shaped lattice below.



8. GORENSTEIN MODELS FOR THE DERIVED CATEGORY AND RECOLLEMENTS

Assume R is any Noetherian ring and let \mathcal{GI} denote the class of Gorenstein injective modules. The authors show in [BGH12] that the Gorenstein injectives are part of an injective cotorsion pair $(\mathcal{W}, \mathcal{GI})$ on $R\text{-Mod}$. The resulting model structure on $R\text{-Mod}$ is called the *Gorenstein injective* model structure on $R\text{-Mod}$ and coincides with the one in [Hov02] when R is a Gorenstein ring. Now applying Theorem 7.2 we potentially get 6 injective model structures on $\text{Ch}(R)$. As illustrated by the examples in the previous section, the number of cotorsion pairs on $\text{Ch}(R)$ induced by $(\mathcal{W}, \mathcal{GI})$ and the canonical $(\mathcal{A}, \mathcal{I})$ increases as we consider more general rings. In particular, when R is non-Gorenstein there are indeed many injective model structures on $\text{Ch}(R)$. Several interesting model structures associated to the canonical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ (even two more than one can obtain from Theorem 7.2) are introduced and discussed in [BGH12]. Three particular model structures of current interest in this setting are the following:

- (1) $\mathcal{M}_1 = (\mathcal{E}, dg\tilde{\mathcal{I}})$ = injective model for the usual derived category.
- (2) $\mathcal{M}_2 = ({}^\perp ex\tilde{\mathcal{I}}, ex\tilde{\mathcal{I}})$ = injective model for the stable derived category.
- (3) $\mathcal{M}_3 = ({}^\perp dw\tilde{\mathcal{I}}, dw\tilde{\mathcal{I}})$ = injective model for the homotopy category of all complexes of injectives (or coderived category in the language of [Pos11]).

These are linked through Krause's recollement [Kra05] indicated below.

$$\begin{array}{ccccc}
 & \xleftarrow{E(\mathcal{M}_2)} & & \xleftarrow{\lambda} & \\
 K_{ex}(Inj) & \xrightarrow{I} & K(Inj) & \xrightarrow{E(\mathcal{M}_3)} & K(DG-Inj) \\
 & \xleftarrow{C(\mathcal{M}_3)} & & \xleftarrow{I} &
 \end{array}$$

Recall that $K(DG-Inj) \cong \mathcal{D}(R)$ and here the notation such as $E(\mathcal{M}_3)$ and $C(\mathcal{M}_3)$ respectively represent using enough injectives or enough projectives with respect to that cotorsion pair. This corresponds to taking special preenvelopes or precovers.

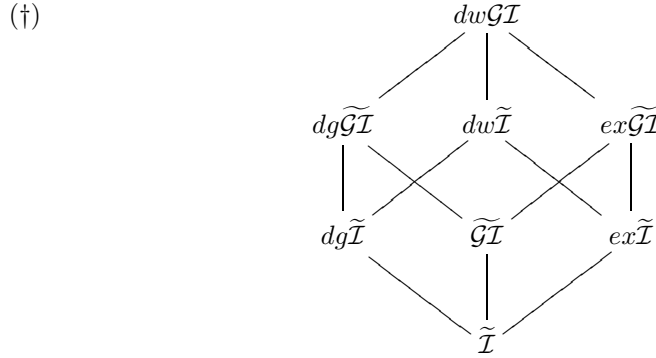
It is not our purpose at this point to make a detailed study of all of the analogous Gorenstein derived categories. We simply wish to point out that the theory here is even richer and that Krause's recollement situation has the three Gorenstein analogs listed in Corollary 8.2 and the Gorenstein projective analogs of Corollary 8.4. In the next section we point out even more recollement situations involving these derived categories. We now set some language following the language set in [BGH12] and referenced in Example 7.6.

- $dw\widetilde{\mathcal{GI}}$ is the class of (categorical) *Gorenstein injective* complexes by Theorem 8.1 below. We call the model structure $({}^\perp dw\widetilde{\mathcal{GI}}, dw\widetilde{\mathcal{GI}})$ the *Gorenstein Inj model structure* on $\text{Ch}(R)$.
- $ex\widetilde{\mathcal{GI}}$ is the class of *exact Gorenstein injective* complexes. We call the model structure $({}^\perp ex\widetilde{\mathcal{GI}}, ex\widetilde{\mathcal{GI}})$ the *exact Gorenstein Inj model structure* on $\text{Ch}(R)$.
- $dg\widetilde{\mathcal{GI}} = \widetilde{\mathcal{W}}^\perp$ is the class of *DG-Gorenstein injective* complexes. We call the model structure $(\widetilde{\mathcal{W}}, dg\widetilde{\mathcal{GI}})$ the *DG-Gorenstein Inj model structure* on $\text{Ch}(R)$.
- $\widetilde{\mathcal{GI}}$ is the class of *exact DG-Gorenstein injective* complexes. We call the model structure $(dg\widetilde{\mathcal{W}}, \widetilde{\mathcal{GI}})$ the *exact DG-Gorenstein Inj model structure* on $\text{Ch}(R)$.

By Proposition 3.8, two chain maps $f, g : X \rightarrow F$ in $\text{Ch}(R)$ (where F is fibrant) are formally homotopic in any of these model structures if and only if their difference factors through an injective complex. But injective complexes are contractible and it follows that the two maps are homotopic if and only if they are chain homotopic in the usual sense. So, for example, the homotopy category of the Gorenstein Inj model structure is equivalent to $K(GInj)$, the chain homotopy category of the Gorenstein injective complexes. Similarly, the homotopy category of the exact Gorenstein Inj model structure will be denoted $K_{ex}(GInj)$. Then we have the DG-versions which we will denote by $K(DG-GInj)$ and $K_{ex}(DG-GInj)$.

Remark 7. We resist any urge to give names to the complexes in $dw\widetilde{\mathcal{W}}$ and $ex\widetilde{\mathcal{W}}$ and $dg\widetilde{\mathcal{W}}$ at this point. However, note the following. Since \mathcal{W} contains all the projective modules, $dw\widetilde{\mathcal{W}}$ (resp. $ex\widetilde{\mathcal{W}}$, resp. $dg\widetilde{\mathcal{W}}$) contains all complexes of projectives (resp. exact complexes of projectives, resp. DG-projective complexes). Moreover, since \mathcal{W} contains the injectives, both $dw\widetilde{\mathcal{W}}$ and $dg\widetilde{\mathcal{W}}$ contain all complexes of injectives and $ex\widetilde{\mathcal{W}}$ contains the exact complexes of injectives.

We have the following portion of the semilattice of injective cotorsion pairs in $\text{Ch}(R)$. But note that we have not even included the model structures corresponding to $dw\widetilde{\mathcal{W}}^\perp$ and $ex\widetilde{\mathcal{W}}^\perp$, which are still interesting especially in light of Theorem 9.1.



8.1. Gorenstein injective complexes. We now wish to characterize the categorical Gorenstein injective complexes, show that these are the fibrant objects in a model structure on $\text{Ch}(R)$ having $\mathcal{D}(R)$ as its homotopy category, and embed these homotopy categories in a Gorenstein version of Krause's recollement.

It has been known for some time that over certain rings, especially Gorenstein rings, that the Gorenstein injective (resp. Gorenstein projective) complexes are precisely those complexes X for which each X_n is Gorenstein injective (resp. Gorenstein projective). For example, see [GR99]. Recently Enochs, Estrada, and Iacob have shown this for Gorenstein projective complexes over a commutative Noetherian ring admitting a dualizing complex [EE05]. Moreover, we see from [YL11] that this is true for any ring R ! But Theorem 5.2 says that the Gorenstein injective complexes sit on the top of the semilattice of injective cotorsion pairs. So the following is a quick and elegant proof of the above fact which works at this point for any Noetherian ring R .

Theorem 8.1. *Let R be any ring for which we know that the Gorenstein injective cotorsion pair $(\mathcal{W}, \mathcal{GI})$ is cogenerated by a set. Then $({}^\perp dw\widetilde{\mathcal{GI}}, dw\widetilde{\mathcal{GI}})$ is also cogenerated by a set and $dw\widetilde{\mathcal{GI}}$ is exactly the class of Gorenstein injective complexes.*

Proof. We choose to prove the projective version this time. See proof of Theorem 8.3. □

Corollary 8.2. *Let R be any Noetherian ring and let $(\mathcal{W}, \mathcal{GI})$ denote the Gorenstein injective cotorsion pair in $R\text{-Mod}$. Then for the three choices of \mathcal{M}_1 and \mathcal{M}_2 as indicated below these are injective cotorsion pairs in $\text{Ch}(R)$ with $\mathcal{M}_2 \preceq_r \mathcal{M}_1$ having right localization $\mathcal{M}_1/\mathcal{M}_2$ a model for the derived category $\mathcal{D}(R)$.*

- (1) $\mathcal{M}_1 = ({}^\perp dw\widetilde{\mathcal{GI}}, dw\widetilde{\mathcal{GI}})$ and $\mathcal{M}_2 = ({}^\perp ex\widetilde{\mathcal{GI}}, ex\widetilde{\mathcal{GI}})$.
- (2) $\mathcal{M}_1 = (\widetilde{\mathcal{W}}, dg\widetilde{\mathcal{GI}})$ and $\mathcal{M}_2 = (dg\widetilde{\mathcal{W}}, \widetilde{\mathcal{GI}})$.
- (3) $\mathcal{M}_1 = (ex\widetilde{\mathcal{W}}, (ex\widetilde{\mathcal{W}})^\perp)$ and $\mathcal{M}_2 = (dw\widetilde{\mathcal{W}}, (dw\widetilde{\mathcal{W}})^\perp)$.

Furthermore, each case gives a recollement with the usual derived category as indicated by Theorem 4.6. For example, the first gives the recollement

$$\begin{array}{ccccc}
 & E(\mathcal{M}_2) & & \lambda & \\
 & \curvearrowright & & \curvearrowright & \\
 K_{ex}(GInj) & \xrightarrow{I} & K(GInj) & \xrightarrow{E(\mathcal{M}_3)} & K(DG-Inj) \\
 & \curvearrowleft & & \curvearrowleft & \\
 & C(\mathcal{M}_3) & & I &
 \end{array}$$

Recall that $K(DG-Inj) \cong \mathcal{D}(R)$ and the notation $E(\mathcal{M}_3)$ and $C(\mathcal{M}_3)$ respectively represent using enough injectives or enough projectives with respect to the cotorsion pair $\mathcal{M}_3 = (\mathcal{E}, dg\widetilde{\mathcal{I}})$. (So taking DG-injective preenvelopes or Exact precovers.)

Proof. We set $\mathcal{M}_3 = (\mathcal{E}, dg\widetilde{\mathcal{I}})$. According to Theorem 4.6, in each case we just need to show $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ and one of the other two conditions. For statements (1) and (2) we will show $\mathcal{E} \cap \mathcal{F}_1 = \mathcal{F}_2$. Then we use the other condition in Theorem 4.6 to prove (3).

Take the first pair, $\mathcal{M}_1 = ({}^\perp dw\widetilde{\mathcal{GI}}, dw\widetilde{\mathcal{GI}})$ and $\mathcal{M}_2 = ({}^\perp ex\widetilde{\mathcal{GI}}, ex\widetilde{\mathcal{GI}})$. They are injective cotorsion pairs in $\text{Ch}(R)$ by Theorem 7.2. Since $\mathcal{F}_1 = dw\widetilde{\mathcal{GI}}$ lies at the top of the semilattice it clearly contains both $\mathcal{F}_2 = ex\widetilde{\mathcal{GI}}$ and $\mathcal{F}_3 = dg\widetilde{\mathcal{I}}$. It is also clear that $\mathcal{E} \cap dw\widetilde{\mathcal{GI}} = ex\widetilde{\mathcal{GI}}$. So we get the recollement for (1).

Now take the second pair, $\mathcal{M}_1 = (\widetilde{\mathcal{W}}, dg\widetilde{\mathcal{GI}})$ and $\mathcal{M}_2 = (dg\widetilde{\mathcal{W}}, \widetilde{\mathcal{GI}})$. They are injective cotorsion pairs in $\text{Ch}(R)$ by Theorem 7.2. We have $\mathcal{E} \cap dg\widetilde{\mathcal{GI}} = \widetilde{\mathcal{GI}}$ since this in general holds from [Gil04]. So all that is left is to show $dg\widetilde{\mathcal{I}} \subseteq dg\widetilde{\mathcal{GI}}$. That is, we wish to show that every DG-injective complex is a DG-Gorenstein injective. Indeed if I is DG-injective then it is injective in each degree, so Gorenstein injective in each degree. Furthermore any map $f : E \rightarrow I$ where E is exact must be null homotopic. In particular, any map $f : W \rightarrow I$ where $W \in \widetilde{\mathcal{W}}$ must be null homotopic. This completes the proof of statement (2).

Next take the third pair, $\mathcal{M}_1 = (ex\widetilde{\mathcal{W}}, (ex\widetilde{\mathcal{W}})^\perp)$ and $\mathcal{M}_2 = (dw\widetilde{\mathcal{W}}, (dw\widetilde{\mathcal{W}})^\perp)$. In this setup we first need to see that the DG-injective complexes are in $(ex\widetilde{\mathcal{W}})^\perp$. But $(ex\widetilde{\mathcal{W}})^\perp$ is the class of all complexes Y of Gorenstein injective modules with the property that any map $f : W \rightarrow Y$ is null homotopic whenever $W \in ex\widetilde{\mathcal{W}}$. So we see that the DG-injectives are in this class. So following the setup to Theorem 4.6 we do have $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ and we will now use the second condition of Theorem 4.6. That is, we will finish by showing $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}_3$. But $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ is clear in this case so let's look at $\mathcal{F}_2 \subseteq \mathcal{W}_3$. It is required to show that every complex in $\mathcal{F}_2 = (dw\widetilde{\mathcal{W}})^\perp$ is exact. But as mentioned in the above Remark, the DG-projective complexes must be in $dw\widetilde{\mathcal{W}}$, and so $(dw\widetilde{\mathcal{W}})^\perp$ must be contained in the class of exact complexes. This completes the proof and we get the three recollements. \square

8.2. The Gorenstein projective complexes. We now look at the dual situation. Here we assume that R is any coherent ring in which each flat module has finite projective dimension. (See Subsection 5.1 for examples of such rings.) With this hypothesis on R we have from [BGH12] that if \mathcal{GP} denotes the class of all Gorenstein projective modules then there is a projective cotorsion pair $(\mathcal{GP}, \mathcal{W})$

which is cogenerated by a set. (The class \mathcal{W} is in general different than the class $\mathcal{W} = {}^\perp \mathcal{GI}$, unless we know R is Gorenstein.) Applying Theorem 7.3 to this projective cotorsion pair results in 6 lifted projective model structures on $\text{Ch}(R)$, all analogous to the Gorenstein injective versions considered above. We record the duals of the injective results, starting with the promised fact that the Gorenstein projective complexes are precisely the complexes having a Gorenstein projective module in each degree.

Theorem 8.3. *Let R be any ring for which we know that the Gorenstein projective cotorsion pair $(\mathcal{GP}, \mathcal{W})$ is cogenerated by a set. Then $(dw\widetilde{\mathcal{GP}}, (dw\widetilde{\mathcal{GP}})^\perp)$ is also cogenerated by a set and $dw\widetilde{\mathcal{GP}}$ is exactly the class of Gorenstein projective complexes.*

Proof. $(dw\widetilde{\mathcal{GP}}, (dw\widetilde{\mathcal{GP}})^\perp)$ is a projective cotorsion pair by Theorem 7.3. So by the “lattice theorem” Theorem 5.4 we have that $dw\widetilde{\mathcal{GP}}$ is contained in the class of all Gorenstein projective complexes. To finish we just need to show that if G is Gorenstein projective then each G_n must be Gorenstein projective. For this suppose G is a Gorenstein projective chain complex so that there is an exact sequence of projective chain complexes

$$\mathcal{P} \equiv \cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow P^{-1} \rightarrow \cdots$$

for which $G = \ker(P^0 \rightarrow P^{-1})$ and which will remain exact after application of $\text{Hom}_{\text{Ch}(R)}(-, Q)$ for any projective chain complex Q . Then for any projective module M taking Q to be the projective complex $D^{n+1}(M)$ we get exactness of the complex $\text{Hom}_{\text{Ch}(R)}(\mathcal{P}, D^{n+1}(M)) \cong \text{Hom}_R(P_n^*, M)$, where P_n^* denotes the complex

$$P_n^* \equiv \cdots \rightarrow P_n^2 \rightarrow P_n^1 \rightarrow P_n^0 \rightarrow P_n^{-1} \rightarrow \cdots$$

Of course P_n^* is an exact complex of projective modules and since we also have $G_n = \ker(P_n^0 \rightarrow P_n^{-1})$ we conclude that G_n is a Gorenstein projective module. \square

Remark 8. We use the notation $\mathcal{M}_2 \backslash \mathcal{M}_1$ from [Bec12] to denote the left localization of a projective cotorsion pair \mathcal{M}_1 by another \mathcal{M}_2 having $\mathcal{C}_2 \subseteq \mathcal{C}_1$. This notation emphasizes a *left* localization.

Corollary 8.4. *Let R be a coherent ring in which all flat modules have finite projective dimension and let $(\mathcal{GP}, \mathcal{W})$ denote the Gorenstein projective cotorsion pair in $R\text{-Mod}$. Then for the three choices of \mathcal{M}_1 and \mathcal{M}_2 as indicated below these are projective cotorsion pairs in $\text{Ch}(R)$ with $\mathcal{M}_2 \preceq_l \mathcal{M}_1$ having left localization $\mathcal{M}_2 \backslash \mathcal{M}_1$ a model for the derived category $\mathcal{D}(R)$.*

- (1) $\mathcal{M}_1 = (dw\widetilde{\mathcal{GP}}, (dw\widetilde{\mathcal{GP}})^\perp)$ and $\mathcal{M}_2 = (ex\widetilde{\mathcal{GP}}, (ex\widetilde{\mathcal{GP}})^\perp)$.
- (2) $\mathcal{M}_1 = (dg\widetilde{\mathcal{GP}}, \widetilde{\mathcal{W}})$ and $\mathcal{M}_2 = (\widetilde{\mathcal{GP}}, dg\widetilde{\mathcal{W}})$.
- (3) $\mathcal{M}_1 = ({}^\perp ex\widetilde{\mathcal{W}}, ex\widetilde{\mathcal{W}})$ and $\mathcal{M}_2 = ({}^\perp dw\widetilde{\mathcal{W}}, dw\widetilde{\mathcal{W}})$.

Furthermore, each case gives a recollement with the usual derived category as indicated by Theorem 4.7. For example, the first gives the recollement

$$\begin{array}{ccccc}
 & E(\mathcal{M}_3) & & I & \\
 & \curvearrowright & & \curvearrowright & \\
 K_{ex}(GProj) & \xrightarrow{I} & K(GProj) & \xrightarrow{C(\mathcal{M}_3)} & K(DG-Proj) \\
 & \curvearrowleft & & \curvearrowleft & \\
 & C(\mathcal{M}_2) & & \rho &
 \end{array}$$

Recall that $K(DG-Proj) \cong \mathcal{D}(R)$ and the notation $E(\mathcal{M}_3)$ and $C(\mathcal{M}_3)$ respectively represent using enough injectives or enough projectives with respect to the cotorsion pair $\mathcal{M}_3 = (dg\tilde{\mathcal{P}}, \mathcal{E})$. (So taking Exact preenvelopes or DG-projective precovers.)

Vista. We can generalize the results of this section to coherent rings R using the notion of the Ding injective modules from [Gil10] and [BGH12]. In more detail, call a module M *Ding injective* if it equals $Z_0 I$ for some exact complex of injectives I which remains exact after applying $\text{Hom}_R(E, -)$ for any FP-injective module E . Then as shown in [BGH12] we have an injective cotorsion pair $(\mathcal{W}, \mathcal{DI})$ where \mathcal{DI} is the class of Ding injectives. There is a dual notion of Ding Projective modules which leads to a projective cotorsion pair $(\mathcal{DP}, \mathcal{V})$. When R is a *Ding-Chen ring* (the coherent analog of a Gorenstein ring) there is a particularly nice balance in that $\mathcal{W} = \mathcal{V}$. Furthermore, when R is Noetherian we get that \mathcal{DI} coincides with the Gorenstein injective modules. On the other hand, when R is such that every flat module has finite projective dimension we get that \mathcal{DP} coincides with the Gorenstein projective modules. So for a Noetherian ring in which all flat modules have finite projective dimension the Ding theory coincides with the Gorenstein theory.

In any case, for a general coherent ring R we may apply Theorem 7.2 to $(\mathcal{W}, \mathcal{DI})$ to get the analogs to the injective cotorsion pairs of chain complexes introduced in this section and Theorem 7.3 to $(\mathcal{DP}, \mathcal{V})$ to get the projective analogs. The analogy is not perfect though. For example, $dw\mathcal{DP}$ is not the class of categorical Ding projective complexes, in general. However, it is shown in [YLL] that $dg\mathcal{DP}$ are the categorical Ding projective complexes and $dg\mathcal{DI}$ are the categorical Ding injective complexes whenever R is a Ding-Chen ring.

9. MORE RECOLLEMENT SITUATIONS

In this section we point out two more interesting recollements arising from the classes of Gorenstein complexes introduced in Section 8. There may be more lurking around as the author has not attempted to systematically find them all. We point out only injective versions, but there are projective versions as well.

We start by noting that since the class $dw\mathcal{GI}$ of Gorenstein injective complexes sits on the top of the semilattice of injective cotorsion pairs, Theorem 4.6 says we get a recollement situation whenever $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$ OR $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}_3$.

Theorem 9.1. *Let R be a Noetherian ring. Then we have the two recollement situations below.*

$$\begin{array}{ccccc}
 & E(\mathcal{M}_2) & & \lambda & \\
 & \curvearrowright & & \curvearrowright & \\
 K(\text{Inj}) & \xrightarrow{I} & K(G\text{Inj}) & \xrightarrow{E(\mathcal{M}_3)} & K((dw\widetilde{\mathcal{W}})^\perp) \\
 & \curvearrowleft & & \curvearrowleft & \\
 & C(\mathcal{M}_3) & & I &
 \end{array}$$

AND

$$\begin{array}{ccccc}
 & E(\mathcal{M}_2) & & \lambda & \\
 & \curvearrowright & & \curvearrowright & \\
 K_{ex}(\text{Inj}) & \xrightarrow{I} & K(G\text{Inj}) & \xrightarrow{E(\mathcal{M}_3)} & K((ex\widetilde{\mathcal{W}})^\perp) \\
 & \curvearrowleft & & \curvearrowleft & \\
 & C(\mathcal{M}_3) & & I &
 \end{array}$$

Proof. Apply Theorem 4.6. For the first, we take $\mathcal{M}_1 = ({}^\perp dw\widetilde{\mathcal{GI}}, dw\widetilde{\mathcal{GI}})$, $\mathcal{M}_2 = ({}^\perp dw\widetilde{\mathcal{I}}, dw\widetilde{\mathcal{I}})$ and $\mathcal{M}_3 = (dw\widetilde{\mathcal{W}}, (dw\widetilde{\mathcal{W}})^\perp)$. It is clear that $dw\widetilde{\mathcal{W}} \cap dw\widetilde{\mathcal{GI}} = dw\widetilde{\mathcal{I}}$ since $\mathcal{W} \cap \mathcal{GI}$ is the class of injective modules.

For the second, we take $\mathcal{M}_1 = ({}^\perp dw\widetilde{\mathcal{GI}}, dw\widetilde{\mathcal{GI}})$, $\mathcal{M}_2 = ({}^\perp ex\widetilde{\mathcal{I}}, ex\widetilde{\mathcal{I}})$ and $\mathcal{M}_3 = (ex\widetilde{\mathcal{W}}, (ex\widetilde{\mathcal{W}})^\perp)$. Again, it is clear that $ex\widetilde{\mathcal{W}} \cap dw\widetilde{\mathcal{GI}} = ex\widetilde{\mathcal{I}}$ since $\mathcal{W} \cap \mathcal{GI}$ is the class of injective modules. \square

Example 9.2. Let R be the quasi-Frobenius ring $\mathbb{Z}/4$ as in Example 7.9. Then the second recollement in Theorem 9.1 above gives a recollement $S(R) \rightarrow K(R) \rightarrow K(\mathcal{W}_{\text{proj}})$ where $S(R) = K_{ex}(\text{Inj})$ is the stable derived category of R , $K(R)$ is the usual homotopy category and $\mathcal{W}_{\text{proj}} = ex\widetilde{\mathcal{P}}^\perp = ex\widetilde{\mathcal{I}}^\perp$ are the trivial objects in the model for the *projective stable derived category*, $S_{\text{proj}}(R)$, of [BGH12]. As shown in [BGH12], we have $\mathcal{W}_{\text{proj}} \neq \mathcal{W}_{\text{inj}}$ but $S_{\text{proj}}(R) \cong S(R)$ through a natural Quillen equivalence. On the other hand, the dual of Theorem 9.1 says there is a recollement $S_{\text{proj}}(R) \rightarrow K(R) \rightarrow K(\mathcal{W}_{\text{inj}})$.

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